

Bandit Learning in General Open Multi-agent Systems

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Abstract

Open systems, where agents arrive and depart over time, are growing in digital platforms. Existing work on online learning in such systems, however, often relies on structural assumptions that may be restrictive in practice. In particular, heterogeneous rewards and general agent patterns create three new challenges: arrivals induce endogenous non-stationarity, agent patterns govern the rate of information accumulation, and uncertainty brought by new agents accumulates over time and scales regret. To this end, we formulate a learning problem for open systems that accommodates both forms of generality. We introduce new concepts to capture the inherent complexities: the *pre-training degree* of new agents quantifies how much information an agent carries upon entry, *stability* measures the impact of population changes on the global optimum, and *global dynamic regret* compares the cumulative expected reward with that of the varying optimal arms. We also study a manifold representation of agents and introduce a *dynamic population embedding* that directly encodes the population functional for each arm. We develop two UCB-based methods. The first combines entry information with online observations through agent-wise confidence bounds and aggregates the estimates and uncertainties for arm selection. The second pools agent-level observations through kernel regression on the manifold and constructs UCBs by projecting the confidence sets onto the current population embedding. Entry uncertainty enters the regret linearly through the pre-training error, while in stable regimes, regret is governed by the time needed to identify the stable optimal arm. With the manifold as a handle, regret is governed by the information gained from agent representations. These bounds all depend on population dynamics, but in different ways.

1 Introduction

Multi-armed bandits [Auer et al., 2002a,b] have been widely studied for their mathematical beauty and broad applicability. Seminal algorithms couple statistical estimation

with uncertainty control [Auer et al., 2002a, Thompson, 1933] and have enabled decision making in e-commerce [Xiang et al., 2022, Jiang et al., 2021, Barraza-Urbina and Glowacka, 2020, Zhu and Van Roy, 2023] and, more recently, large language models (LLMs) [Atalar, 2026, Poon et al., 2026, Bouneffouf and Feraud, 2026, Bouneffouf and Féraud, 2024]. However, modern systems increasingly involve multiple decision-makers, ranging from distributed services [Zhu et al., 2021, Yang et al., 2024] and store branches [Mes and Gerrits, 2018, Khatua et al., 2011] to teams of agents in agentic systems [He et al., 2025, Chen et al., 2025]. The classical bandit framework no longer directly applies, and multi-agent multi-armed bandits (MA-MABs) emerge as a recipe [Landgren et al., 2016a,b, 2021, Zhu et al., 2020, Martínez-Rubio et al., 2019, Agarwal et al., 2022, Wang et al., 2022, 2020, Li and Song, 2022, Sankararaman et al., 2019, Chawla et al., 2020, Xu and Klabjan, 2023, 2024, 2025]. In MA-MABs, agents are nodes on a graph. At each round, each agent selects an action, observes the local reward of the chosen arm, communicates with its neighbors, and incorporates the received information into its decisions. The objective is to always select the arm with the highest aggregate reward over all agents.

To date, it is habitually assumed that the system is closed, in the sense that the set of decision-makers remains fixed throughout. In practice, however, data centers and their tenants may expand or shut down [Shehabi et al., 2018, Shieh et al., 2011], store branches may open or close [Clapp et al., 2014], and LLM agents may be added to or removed from agentic workflows [Liu et al., 2026a, Wang et al., 2025]. Such systems have an evolving set of decision-makers and are often described as open multi-agent systems (OMAS) [Deplano et al., 2026, Liu et al., 2026b]. Recent work has begun to study learning therein [Rosenski et al., 2016, Trinh and Combes, 2021], assuming that active agents are drawn from a fixed pool of bounded size. This may not apply when populations can grow arbitrarily. [Xu and Klabjan, 2026] removes this restriction via Poisson processes, assuming homogeneous rewards across agents. However, both assumptions can still be restrictive: rewards for the same arm may vary across agents [Yang et al., 2021], while population changes may again be arbitrary [Deplano et al., 2026].

Removing them, yet, changes both the objective and the learning process. As rewards are heterogeneous, an arriving or departing agent changes the aggregate reward of each arm by a different amount, while population dynamics determine when and how strongly these changes affect the system. Thus, even when every agent has stationary rewards, changes in the active population can change the globally optimal arm. A fixed global benchmark appears inadequate. In the meantime, an arriving agent may bring only limited information about its own rewards, while information learned from existing agents may not transfer to it under heterogeneity. The less accurate an arriving agent’s prior information is, the more uncertainty it introduces into the learning process. Population dynamics further determine how much such perturbations accumulate.

From this interaction, open systems can exhibit different intrinsic difficulty. In the worst case, each arrival can introduce enough uncertainty or shift the aggregate rewards enough to create a new learning problem. In more favorable regimes, the active population is large enough that later arrivals have only transient effects and the optimal arm eventually stabilizes. Also favorably and more generally, agents may share latent structures that allow information to transfer across them, so that what matters for decision making is easier to learn than their separate rewards. All in all, the difficulty depends on which aspects of openness are key to the decision making. Hence, we hope to know:

Can we disentangle the sources of learning difficulty and address them in open systems?

Main Contributions. We evaluate an arm through its population-level value, obtained by aggregating its rewards over the current population, and target the arm with the largest such value, yielding global dynamic regret. This benchmark exposes two effects of openness. Arrivals introduce uncertainty before observations are collected, which we quantify through the *pre-training error* P_m . Population changes can also alter which arm is globally optimal, motivating our notion of *stability*.

We develop methods that target different sources of learning difficulty in open systems. Agent-Wise Global-UCB combines an arriving agent’s entry estimate with observations collected after entry. For each agent and arm, it conditionally uses the entry estimate or the empirical estimate according to which has the smaller confidence radius, and then aggregates these conditional estimates and their uncertainties for arm selection. Motivated by the insight that decisions depend only on aggregate rewards, we represent agents on a common manifold, model each arm by a reward function in an RKHS, and use a dynamic population embedding to express its aggregate reward as a population functional. We estimate the reward functions through kernel ridge regression and project their estimates and confidence sets onto the dynamic population embedding. In this way, we directly estimate the aggregate reward and control its uncertainty. We further enable parallel exploration by adapting a top-two sampling principle. The arm with the largest UCB serves as the leader, while its plausible competitors form the challenger set; exploratory agents are then assigned to challengers via matching to maximize the reduction in uncertainty.

For Agent-Wise Global-UCB, regret decomposes into the cumulative pre-training error and the post-entry learning cost, which depends on their lifetimes and the statistical rate. These dependencies are unavoidable on hard instances, up to logarithmic factors. With a persistent globally optimal arm, it admits $O(\sum_{t=1}^{\tau} M_t)$, where τ is the burn-in time. Alongside manifold-based learning, we find two complexities that jointly govern regret. The cumulative information gain Γ_T^{Σ} measures the information accumulated about the reward functions, while the population path $H_T = \sum_{t=1}^T M_t^2 / \log(1+M_t)$ quantifies the amplification of uncertainty induced by population dynamics. Their interaction yields

$O(\beta_T \sqrt{\Gamma_T^\Sigma H_T})$. Under a Matérn kernel, it yields $\tilde{O}(K^{\nu/(2\nu+d)} T^{1/2+d/(2\nu+d)})$, determined by the manifold dimension d and smoothness ν . Parallel exploration introduces a trade-off in the regret bound. It reduces the amplification due to population size, at the cost of extra exploration for information acquisition. Whether this improves the regret bound depends on whether the uncertainty reduction outweighs this cost, which in turn depends on the matching.

2 Problem Formulation

We study learning in an open system with K arms, indexed by $[K] = \{1, 2, \dots, K\}$, over a horizon of T rounds, indexed by $t \in [T]$. We use $t = 0$ to denote the initial system state. At each round t , agents may arrive or depart, yielding the active set $\mathcal{M}_t$. The active agents then select arms, observe their local rewards, and exchange information via the communication graph $G_t = (\mathcal{V}_t, \mathcal{E}_t)$, where $\mathcal{V}_t = \mathcal{M}_t$. Thereafter, we assume complete communication among active agents to focus on the challenges induced by reward heterogeneity and population dynamics. For each $m \in \mathcal{M}_t$, let $\mathcal{N}_m(t) = \{n \in \mathcal{M}_t : (m, n) \in \mathcal{E}_t\}$ denote its neighbor set.

Agents may arrive and depart according to general patterns herein. We denote $\mathcal{A}_t = \mathcal{M}_t \setminus \mathcal{M}_{t-1}$ and $\mathcal{D}_t = \mathcal{M}_{t-1} \setminus \mathcal{M}_t$ as the sets of arrivals and departures. A common special case is Poisson dynamics [Birnbaum, 1954, Xu and Klabjan, 2026], under which the numbers of arrivals and departures at round t follow $A_t = |\mathcal{A}_t| \sim \text{Poisson}(\lambda_A)$ and $D_t = |\mathcal{D}_t| \sim \min\{|\mathcal{M}_t|, \text{Poisson}(\lambda_D)\}$, respectively. The population size then evolves as $M_t = |\mathcal{M}_t| = (M_{t-1} + A_t - D_t)_+$, starting from M_0 . Setting $\lambda_A = \lambda_D = 0$ degenerates to fixed populations. We only require $M_t \geq 1$ throughout.

If agent $m \in \mathcal{M}_t$ pulls arm i at round t , it receives reward $0 \leq r_m^i(t) \leq 1$ from a stationary distribution with mean $0 \leq \mu_m^i = \mathbb{E}[r_m^i(t)] \leq 1$. Different from the homogeneous setting [Xu and Klabjan, 2026], we allow $\mu_m^i \neq \mu_n^i$ for $m \neq n$; the homogeneous model is recovered when $\mu_m^i = \mu^i$ for all m and i . As is standard in the bandit literature, we assume that the rewards are conditionally and independently R -sub-Gaussian given the mean values. To evaluate an arm at the system level, we aggregate its mean rewards over the active agents. Precisely, for arm i , let us define the aggregate reward $V_t(i) = \sum_{m \in \mathcal{M}_t} \mu_m^i$, and its population average $\bar{V}_t(i) = \frac{1}{M_t} \sum_{m \in \mathcal{M}_t} \mu_m^i$, where the latter offsets the effect of population size.

Pre-training degree/error. A key feature of open systems is that an arriving agent may already carry an initial estimator that is informative about its local rewards. For an agent $m \in \mathcal{A}_t$, and let $\hat{\mu}_i^m(t)$ denote the arm- i estimator available to agent m upon entry. This estimator may come from pre-training, side information, or inherited knowledge. We define the *pre-training degree* and *error* of agent m by $D_m = 1/P_m$ and $P_m = \max_{i \in [K]} |\hat{\mu}_i^m(T_m^a - 1) - \mu_m^i|$; $D_m = \infty$ when $P_m = 0$. Thus, P_m measures the

worst-case initialization error of agent m across arms.

Remark. *The notion of pre-trained agents in MA-MAB characterizes the accuracy and flow of prior information in open systems. Inspired by pre-trained models in the LLM community, it may also provide a new route for extending MA-MABs to systems of LLM agents. The concept of pre-training error is also related to the information-theoretic view of nonstationary bandits in [Min and Russo, 2023], although the source of nonstationarity differs, which naturally leads to different measures of its complexity. In their setting, the entropy rate of the optimal-action process measures how difficult it is to track a changing environment, while the information ratio characterizes the cost of acquiring the information needed to do so. In our setting, nonstationarity arises endogenously as arrivals change the aggregate reward vector. The cumulative pre-training error measures how such uncertainty accumulates over time.*

We then quantify how much the global reward of each arm changes as the population evolves through $\max_{i \in [K]} |V_t(i) - V_{t-1}(i)|$ and $\max_{i \in [K]} |\bar{V}_t(i) - \bar{V}_{t-1}(i)|$. Since M_t is common across arms at round t , $V_t(i)$ and $\bar{V}_t(i)$ have the same maximizer. Let $i_t^* \in \arg \max_{i \in [K]} \bar{V}_t(i)$. We track whether the globally optimal arm changes from round $t-1$ to round t through the *stability* indicator $S_t = \mathbf{1}\{i_t^* = i_{t-1}^*\}$. This notion uniquely belongs to the interplay between openness and reward heterogeneity. In a closed system, the population is fixed; in a homogeneous open system, population changes scale all rewards in the same way. Either way, the identity of i_t^* is time-invariant and $S_t \equiv 1$.

Objective. We express system performance in our go-to language of *regret*. Let $a_m(t) \in [K]$ denote the arm selected by agent m at round t . Compared to i_t^* , the instantaneous regret of its choice $a_m(t)$ is $\bar{V}_t(i_t^*) - \bar{V}_t(a_m(t))$. Summing these losses across all active agents and rounds yields the *global dynamic regret*

$$R_T = \sum_{t=1}^T \sum_{m \in \mathcal{M}_t} [\bar{V}_t(i_t^*) - \bar{V}_t(a_m(t))].$$

Here R_T retains the effect of population size and the *normalized regret*

$$\bar{R}_T = \sum_{t=1}^T \frac{1}{M_t} \sum_{m \in \mathcal{M}_t} [\bar{V}_t(i_t^*) - \bar{V}_t(a_m(t))]$$

removes this scale. Notably, when all select the same arm a_t , the two quantities simplify to $R_T = \sum_{t=1}^T [V_t(i_t^*) - V_t(a_t)]$ and $\bar{R}_T = \sum_{t=1}^T [\bar{V}_t(i_t^*) - \bar{V}_t(a_t)]$.

3 Methodology

We first present a relatively direct method, agent-wise Global-UCB, which learns the mean rewards of individual agents and *controls their uncertainty*, and applies when

new agents enter with reward estimates of quantifiable accuracy. We then introduce a manifold-based method that exploits geometric structure across agents and directly learns the aggregate reward of the population and more importantly, *controls its uncertainty*.

3.1 Agent-wise Learning

The first method (Algorithm 1) learns the agent-wise rewards by combining information available when an agent enters with observations collected after entry. Let us index distinct agents by their order of first appearance, and let $j(m)$ denote the index of agent m . Such an agent carries an entry estimate $\hat{\mu}_{m,i}^{\text{entry}}$ with a pretraining error $\bar{P}_m$ satisfying $\max_{i \in [K]} |\hat{\mu}_{m,i}^{\text{entry}} - \mu_m^i| \leq \bar{P}_m$.

For $n \geq 1$ and confidence parameter $\delta \in (0, 1)$, let us define the empirical confidence radius $c_m(n) = \sqrt{\frac{1}{2n} \log(\frac{2Kj(m)(j(m)+1)n(n+1)}{\delta})}$; $c_m(0) = \infty$. For each continuing agent $m \in \mathcal{R}_t$ with $\mathcal{R}_t = \mathcal{M}_t \cap \mathcal{M}_{t-1}$ and arm i , let $n_{m,i}(t-1)$ and $\bar{r}_{m,i}(t-1)$ denote its number of observations and empirical mean. We set $\rho_{m,i}(t-1) = \min\{\bar{P}_m, c_m(n_{m,i}(t-1))\}$, and use $\hat{\mu}_{m,i}(t-1) = \begin{cases} \hat{\mu}_{m,i}^{\text{entry}}, & \bar{P}_m \leq c_m(n_{m,i}(t-1)), \\ \bar{r}_{m,i}(t-1), & \bar{P}_m > c_m(n_{m,i}(t-1)). \end{cases}$ Through this, we use the entry estimate until the observations collected yield a tighter confidence radius.

At round t , the aggregate expected reward $V_t(i) = \sum_{m \in \mathcal{M}_t} \mu_m^i$ is estimated by $\hat{V}_t(i) = \sum_{m \in \mathcal{R}_t} \hat{\mu}_{m,i}(t-1) + \sum_{m \in \mathcal{A}_t} \hat{\mu}_{m,i}^{\text{entry}}$. We compute the confidence radius for agents in $\mathcal{R}_t$ and $\mathcal{A}_t$ as their vanilla sums, i.e., $B_t^{\text{stat}}(i) = \sum_{m \in \mathcal{R}_t} \rho_{m,i}(t-1)$ and $E_t^A = \sum_{m \in \mathcal{A}_t} \bar{P}_m$.

The total confidence width should naturally add uncertainty from new arrivals, E_t^A , to that of continuing agents, $B_t^{\text{stat}}(i)$. Since E_t^A is common across arms by definition, we define the UCB index as $\hat{V}_t(i) + B_t^{\text{stat}}(i)$, and select $a_t \in \arg \max_{i \in [K]} \{\hat{V}_t(i) + B_t^{\text{stat}}(i)\}$. All active agents share the same UCB index and therefore select the common arm a_t . Each agent then observes its local reward and updates the empirical mean and pull count of the selected arm.

If the same arm remains globally optimal with a persistent gap, we say that the system has a stable regime. In this regime, once the uncertainty maintained by the policy becomes and remains sufficiently small w.r.t. the gap, the stable optimal arm has the largest UCB index at every subsequent round and is selected thereafter. The resulting behavior resembles explore-then-commit. Although exploration is driven adaptively by the UCB indices, not by a prescribed exploration phase, the policy eventually enters an analogous commitment phase.

3.2 Manifold-based Learning

A potential pain point of the preceding method is its reliance on a “good enough” pre-training error, which may be out of reach in practice. For example, the pre-training error may be unavailable, difficult to quantify a priori, or just too large to provide useful information. It can then propagate to the population level, where the decision becomes more delicate. However, ultimately, decisions depend on their aggregate rewards over the population, not necessarily requiring control of agent-wise reward uncertainty. This motivates us to *exploit the structure of agent rewards and work directly with uncertainty in the aggregate rewards*. Although arrivals and departures change the active agents and their reward vectors, observations can transfer across agents when they share a common geometric structure. Our handle is to represent agents on a manifold, learn the reward function of each arm over this manifold, and express its decision-relevant uncertainty through a functional induced by the population.

Let $\mathcal{X}$ be a known compact Riemannian manifold. Each agent m has an observed representation $x_m \in \mathcal{X}$, available upon entry and fixed during its lifetime. For each arm $i \in [K]$, let $f_i : \mathcal{X} \rightarrow [0, 1]$ denote its unknown reward function, so that $\mu_m^i = f_i(x_m)$. Let $k : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ be a known positive-definite kernel with associated RKHS $\mathcal{H}_k$, and we define the canonical feature embedding as $\phi(x) = k(x, \cdot) \in \mathcal{H}_k$. As is standard, we assume $0 < k(x, x) \leq 1$ for every x and $\|f_i\|_{\mathcal{H}_k} \leq S$ for every arm i . By the reproducing property of the RKHS, $f_i(x) = \langle f_i, \phi(x) \rangle_{\mathcal{H}_k}$. In other words, the embedding connects the location of agent m on the manifold to its reward.

We aggregate the feature embeddings of the active agents into $\Psi_t = \sum_{m \in \mathcal{M}_t} \phi(x_m) \in \mathcal{H}_k$. We call Ψ_t the *dynamic population embedding*. By linearity and the reproducing property (also the Riesz representation theorem [Goodrich, 1970]), we have that

$$V_t(i) = \sum_{m \in \mathcal{M}_t} f_i(x_m) = \sum_{m \in \mathcal{M}_t} \langle f_i, \phi(x_m) \rangle_{\mathcal{H}_k} = \langle f_i, \Psi_t \rangle_{\mathcal{H}_k}.$$

Hence, for evaluating arm i , it is enough to learn how f_i behaves on the current embedding Ψ_t .

For each i , we estimate f_i using kernel ridge regression:

$$\hat{f}_{i,t-1} \in \arg \min_{f \in \mathcal{H}_k} \left\{ \sum_{\substack{s < t: \\ a_s = i}} \sum_{m \in \mathcal{M}_s} (r_m^i(s) - f(x_m))^2 + \|f\|_{\mathcal{H}_k}^2 \right\}.$$

Its normal equation is $\mathcal{A}_{i,t-1} \hat{f}_{i,t-1} = b_{i,t-1}$, where $\mathcal{A}_{i,t-1} = I + \sum_{s < t: a_s = i} \sum_{m \in \mathcal{M}_s} \phi(x_m) \otimes \phi(x_m)$ and $b_{i,t-1} = \sum_{s < t: a_s = i} \sum_{m \in \mathcal{M}_s} r_m^i(s) \phi(x_m)$, with $(u \otimes u)h = \langle h, u \rangle_{\mathcal{H}_k} u$.

The estimate of the current aggregate reward is then expressed as $\hat{f}_{i,t-1} = \mathcal{A}_{i,t-1}^{-1} b_{i,t-1}$ and $\hat{V}_t(i) = \langle \hat{f}_{i,t-1}, \Psi_t \rangle_{\mathcal{H}_k}$. Hence, the decision rule uses the estimate only through

its projection onto the dynamic population embedding. But more than that, we need to quantify how much uncertainty within the estimate. Precisely, for $n \geq 0$, let $\gamma_n = \sup_{x_1, \dots, x_n \in \mathcal{X}} \frac{1}{2} \log \det(I_n + K_n)$, where K_n is the Gram matrix of $\{x_1, \dots, x_n\}$. Intuitively, γ_n quantifies the maximum information about the reward function that one hopes to obtain from observations at n agent representations; it naturally connects to mutual information as seen later. Let $\bar{\gamma}_n$ be any known nondecreasing sequence satisfying $\gamma_n \leq \bar{\gamma}_n$; the choice of this sequence depends on the kernel $k(\cdot, \cdot)$.

For arm i , let $N_i(t-1) = \sum_{\substack{s \leq t \\ a_s = i}} M_s$ denote the number of observations collected by round $t-1$. For confidence level $\delta \in (0, 1)$, let us define the exploration parameter as $\beta_{i,t-1} = S + R\sqrt{2(\bar{\gamma}_{N_i(t-1)} + 1 + \log \frac{K}{\delta})}$ and the uncertainty in the aggregate reward $V_t(i)$ as $\sigma_{i,t}^{\text{pop}} = \sqrt{\langle \Psi_t, \mathcal{A}_{i,t-1}^{-1} \Psi_t \rangle_{\mathcal{H}_k}}$. As shown in Section 4.2, w.p. at least $1 - \delta$, $|\hat{V}_t(i) - V_t(i)| \leq \beta_{i,t-1} \sigma_{i,t}^{\text{pop}}$ simultaneously over all arms and rounds. Each agent selects $a_t \in \arg \max_{i \in [K]} \{\hat{V}_t(i) + \beta_{i,t-1} \sigma_{i,t}^{\text{pop}}\}$ following the UCB principle. We then observe a batch of agent-level rewards and update $\mathcal{A}_{a_t,t}$ and $b_{a_t,t}$.

Finite-dimensional implementation. Although the method is formulated in the possibly infinite-dimensional RKHS $\mathcal{H}_k$, all decision-relevant quantities admit finite-dimensional kernel representations.

Let us enumerate the $N_i(t-1)$ observations of arm i as $\{(x_{i,\ell}, y_{i,\ell})\}_{\ell=1}^{N_i(t-1)}$, and let $y_{i,t-1} \in \mathbb{R}^{N_i(t-1)}$ collect their rewards. Let us consider an operator which evaluates the values of any function in RKHS w.r.t. observation x as $\Phi_{i,t-1} : \mathcal{H}_k \rightarrow \mathbb{R}^{N_i(t-1)}$, $[\Phi_{i,t-1} f]_\ell = f(x_{i,\ell})$, with adjoint $\Phi_{i,t-1}^* a = \sum_{\ell=1}^{N_i(t-1)} a_\ell \phi(x_{i,\ell})$, $a \in \mathbb{R}^{N_i(t-1)}$. The Gram matrix is then $K_{i,t-1} = \Phi_{i,t-1} \Phi_{i,t-1}^*$. For any $f \in \mathcal{H}_k$, by the reproducing property, we obtain

$$\Phi_{i,t-1}^* \Phi_{i,t-1} f = \sum_{\ell=1}^{N_i(t-1)} (\phi(x_{i,\ell}) \otimes \phi(x_{i,\ell})) f.$$

Hence, we derive that $\mathcal{A}_{i,t-1} = I + \Phi_{i,t-1}^* \Phi_{i,t-1}$. Next we proceed to the representation of $b_{i,t-1}$. By the definition of the adjoint, $\Phi_{i,t-1}^* y_{i,t-1} = \sum_{\ell=1}^{N_i(t-1)} y_{i,\ell} \phi(x_{i,\ell}) = b_{i,t-1}$. With these equivalences, we obtain $\hat{f}_{i,t-1} = \left(I + \Phi_{i,t-1}^* \Phi_{i,t-1}\right)^{-1} \Phi_{i,t-1}^* y_{i,t-1}$. Using $\left(I + \Phi_{i,t-1}^* \Phi_{i,t-1}\right)^{-1} \Phi_{i,t-1}^* = \Phi_{i,t-1}^* \left(I + \Phi_{i,t-1} \Phi_{i,t-1}^*\right)^{-1}$, we obtain $\hat{f}_{i,t-1} = \Phi_{i,t-1}^* (K_{i,t-1} + I)^{-1} y_{i,t-1}$. Let $q_{i,t} = \Phi_{i,t-1} \Psi_t$ which collects the values of the population embedding Ψ_t at the agent representation histories $\{x_{i,\ell}\}$. Its ℓ -th entry is $[q_{i,t}]_\ell = \Psi_t(x_{i,\ell}) = \sum_{m \in \mathcal{M}_t} k(x_{i,\ell}, x_m)$. Moreover, we denote c_t as the squared norm of the population embedding Ψ_t as $c_t = \|\Psi_t\|_{\mathcal{H}_k}^2 = \sum_{m,n \in \mathcal{M}_t} k(x_m, x_n)$. Consequently, we

obtain

$$\begin{aligned}\widehat{V}_t(i) &= \langle \widehat{f}_{i,t-1}, \Psi_t \rangle_{\mathcal{H}_k} = \langle \Phi_{i,t-1}^* (K_{i,t-1} + I)^{-1} y_{i,t-1}, \Psi_t \rangle_{\mathcal{H}_k} \\ &= q_{i,t}^\top (K_{i,t-1} + I)^{-1} y_{i,t-1}.\end{aligned}$$

Using the Woodbury identity matrix [Henderson and Searle, 1981] yields $\mathcal{A}_{i,t-1}^{-1} = I - \Phi_{i,t-1}^* (K_{i,t-1} + I)^{-1} \Phi_{i,t-1}$. Hence, we express the radius as

$$\begin{aligned}(\sigma_{i,t}^{\text{pop}})^2 &= \langle \Psi_t, \mathcal{A}_{i,t-1}^{-1} \Psi_t \rangle_{\mathcal{H}_k} \\ &= \|\Psi_t\|_{\mathcal{H}_k}^2 - (\Phi_{i,t-1} \Psi_t)^\top (K_{i,t-1} + I)^{-1} (\Phi_{i,t-1} \Psi_t) \\ &= c_t - q_{i,t}^\top (K_{i,t-1} + I)^{-1} q_{i,t}.\end{aligned}$$

In other words, the kernel representation based on past observations is algebraically equivalent to the infinite-dimensional counterpart above and therefore yields the same UCB indices and arm choices, while providing a more computationally appealing form. The full procedure is exhibited in Algorithm 2.

3.2.1 Parallel Exploration

Thus far, all active agents select a common arm at each round. This idea naturally aligns with the system-level objective and is feasible under the complete communication graph. Under the manifold formulation, an observation from one agent can reduce uncertainty in the aggregated reward, yet the extent of this reduction varies across agents. This heterogeneity in information value creates an organic opportunity for parallel exploration. To this end, we next allow agents to diversify arms while aiming at the same target $i_t^* \in \arg \max_{i \in [K]} V_t(i)$. This creates extra freedom for information collection, in that agents can explore arms in parallel to jointly inform the decision, but of course as we shall foresee shortly, also an extra layer of complexity in exploration.

For each arm i , let $\mathcal{Q}_{t,i} = \{m \in \mathcal{M}_t : a_m(t) = i\}$ denote the active agents assigned to arm i , and let $M_{t,i} = |\mathcal{Q}_{t,i}|$. Then $\{\mathcal{Q}_{t,i}\}_{i \in [K]}$ partitions $\mathcal{M}_t$, with $\sum_{i=1}^K M_{t,i} = M_t$. Likewise, observations are accumulated from agents assigned to that arm, so we can equivalently write $N_i(t-1) = \sum_{s < t} M_{s,i}$, $\mathcal{A}_{i,t-1} = I + \sum_{s < t} \sum_{m \in \mathcal{Q}_{s,i}} \phi(x_m) \otimes \phi(x_m)$ and $b_{i,t-1} = \sum_{s < t} \sum_{m \in \mathcal{Q}_{s,i}} r_m^i(s) \phi(x_m)$. The estimator $\widehat{f}_{i,t-1} = \mathcal{A}_{i,t-1}^{-1} b_{i,t-1}$ induces the estimate $\widehat{V}_t(i) = \langle \widehat{f}_{i,t-1}, \Psi_t \rangle_{\mathcal{H}_k}$ and uncertainty $\sigma_{i,t}^{\text{pop}}$. Here the confidence bound continues to hold, i.e., w.h.p. $|\widehat{V}_t(i) - V_t(i)| \leq \beta_{i,t-1} \sigma_{i,t}^{\text{pop}}$.

We adapt top-two sampling to handle parallel exploration. Let us define the upper and lower confidence bounds (UCB and LCB, respectively) of arms as

$$U_t(i) = \widehat{V}_t(i) + \beta_{i,t-1} \sigma_{i,t}^{\text{pop}}, \quad L_t(i) = \widehat{V}_t(i) - \beta_{i,t-1} \sigma_{i,t}^{\text{pop}}.$$

This leads us to the top-two arms as

$$h_t \in \arg \max_{i \in [K]} U_t(i), \quad \mathcal{C}_t = \{i \neq h_t : U_t(i) \geq L_t(h_t)\}.$$

Intuitively, the top-one arm h_t maximizes the UCB index and is therefore a promising candidate for the best arm, while the *arm set* $\mathcal{C}_t$ contains arms whose UCB indices exceed the LCB of h_t , and are thus also plausible candidates for the best arm. We retain h_t (leader) and $\mathcal{C}_t$ (challengers) to promote exploitation and exploration, respectively.

We next assign agents to either the leader or its plausible challengers. Let the exploration budget $e_t = \min\{M_t - 1, |\mathcal{C}_t|\}$, so that at least one agent selects h_t . When $\mathcal{C}_t = \emptyset$, $e_t = 0$ and all agents select h_t . Suppose agent m were assigned to arm i . Its observation would add the rank-one term $\phi(x_m) \otimes \phi(x_m)$ to the current design operator $\mathcal{A}_{i,t-1}$, yielding $\mathcal{A}_{i,t-1}^{(m)} = \mathcal{A}_{i,t-1} + \phi(x_m) \otimes \phi(x_m)$. Correspondingly, we obtain $(\sigma_{i,t}^{\text{pop},(m)})^2 = \langle \Psi_t, (\mathcal{A}_{i,t-1}^{(m)})^{-1} \Psi_t \rangle_{\mathcal{H}_k}$. Thus, the uncertainty reduction in $V_{i,t}$ from this assignment is $G_{i,t}(m) = (\sigma_{i,t}^{\text{pop}})^2 - (\sigma_{i,t}^{\text{pop},(m)})^2$. By the Sherman-Morrison identity [Sherman and Morrison, 1950],

$$G_{i,t}(m) = \langle \Psi_t, \mathcal{A}_{i,t-1}^{-1} \phi(x_m) \rangle_{\mathcal{H}_k}^2 / (1 + \langle \phi(x_m), \mathcal{A}_{i,t-1}^{-1} \phi(x_m) \rangle_{\mathcal{H}_k}).$$

We select e_t agent-arm pairs $\mathcal{E}_t \subseteq \mathcal{M}_t \times \mathcal{C}_t$ by maximizing the total uncertainty reduction, $W_t(\mathcal{E}_t) = \sum_{(m,i) \in \mathcal{E}_t} G_{i,t}(m)$. This is a bipartite matching problem, which can be solved independently as a network flow problem [Goldberg et al., 1989]. Every agent not included in the matching $\mathcal{E}_t$ selects h_t .

4 Theoretical Analyses

4.1 Agent-wise Global UCB

We first characterize the regret bound under Algorithm 1, which seeks to estimate the agent-wise mean rewards. For each agent m , let $N_{m,i}^R(T) = \sum_{t=1}^T \mathbf{1}\{m \in \mathcal{R}_t, a_t = i\}$, and $L_m = \sum_{i=1}^K N_{m,i}^R(T)$, which denote the total number of post-entry observations collected from agent m for arm i and across all arms, respectively. In other words, L_m is the lifetime of agent m . For any confidence level $0 < \delta \leq 1$ and integer n , let us define $\Lambda_m(n) = \log(2Kj(m)(j(m)+1)n(n+1)/\delta)$.

Theorem 1. *With Algorithm 1, with probability at least $1 - \delta$,*

$$R_T \leq 2 \sum_{t=1}^T \sum_{m \in \mathcal{A}_t} \bar{P}_m + 2 \sum_{m: L_m > 0} \min \left\{ L_m \bar{P}_m, K \bar{P}_m + \sqrt{2K \Lambda_m(L_m) L_m} \right\} \quad (1)$$

The bound reveals two complementary sources of complexity. New agents always bring in error P_m , amplified by $\mathcal{A}_t$ over time upon entry due to reward heterogeneity.

During the learning process, when the initial reward estimate is sufficiently accurate, its contribution to regret is further controlled by $L_m \bar{P}_m$. Otherwise, observations collected after entry progressively refine the estimate, leading to the statistical term $\sqrt{K \Lambda_m(L_m) L_m}$. Population dynamics affect the bound through both the agents that enter the system $\mathcal{A}_t$ and each agent's lifetime L_m . The exact order depends on $\hat{\mu}_{m,i}^{\text{entry}}$ and $\bar{P}_m$, which arise in common models such as linear and cluster models; we refer to Table 1 in Appendix A.2.

Indeed, the linear dependence on the pre-training error, the first term in (1), is unavoidable in a heterogeneous open system. In a closed system, $|\mathcal{A}_t| = 0$, and in the worst case, $\bar{P}_m = 1$, so the first term does not enter the rate, and the dominant term in the regret bound is $O(\sqrt{T})$, consistent with the current literature. In a homogeneous open system, $\bar{P}_m$ is the current confidence radius shared by all agents, which shrinks over time. With proper sampling that pools observations across agents in the algorithmic step to optimize sample complexity, the algorithm may recover the rate in [Xu and Klabjan, 2026]. The second term captures the post-entry learning cost for continuing agents. It cannot be removed, as it matches the $O(\sqrt{KT})$ order in closed systems and also holds in general up to a logarithmic factor in T . To see why, we consider two distinct hard instances. First, newly arriving agents may remain for only one round, so their rewards affect the current decision before any online observations are available. Second, agents may appear in disjoint episodes with unrelated reward vectors, forcing the learner to solve a fresh bandit problem after each arrival. The theorem formalizes the idea.

Theorem 2. *Let us assume $K = 2$ and a sequence of arrivals $\{\mathcal{A}_t\}_{t=1}^T$ and numbers $\{p_m \in [0, 1] : m \in \mathcal{A}_t, t \in [T]\}$. There exists instances $\mathcal{I}(\{p_m\})$ with $P_m = p_m$ and instances $\mathcal{Q}(\{L_m, p_m\})$ with $\sum_m (L_m + 1) \leq T$. For any random policy π with coordinated actions, $\sup_{\nu \in \mathcal{I}(\{p_m\})} \mathbb{E}_\nu^\pi[R_T] \geq 1/2 \sum_{t=1}^T \sum_{m \in \mathcal{A}_t} \bar{P}_m$ and $\sup_{\nu \in \mathcal{Q}(\{L_m, p_m\})} \mathbb{E}_\nu^\pi[R_T] \geq 1/128 \sum_{m: L_m > 0} \min\{L_m p_m, \sqrt{K L_m}\}$.*

4.1.1 Sharper Bounds under a Stable Arm

Although agents may be heterogeneous, the aggregate population reward admits a persistent optimal arm. This can arise, for example, when the population is sufficiently large relative to new arrivals after certain time steps and the separation between arms is sufficiently large that the additional reward heterogeneity introduced by new arrivals does not overturn the optimal arm. We formalize this favorable regime as the following assumption on the problem instance.

Assumption 1. *There exist $i^\dagger \in [K]$, $\Delta > 0$, and $\tau_{\text{stab}} \leq T$ s.t. $\bar{V}_t(i^\dagger) - \max_{i \neq i^\dagger} \bar{V}_t(i) \geq \Delta$, $\forall t \geq \tau_{\text{stab}}$.*

However, the existence of such an arm is only part of the story, as the performance also depends on whether the method is able to identify this arm, which is characterized by another burn-in period. Let $W_t(i) = B_t^{\text{stat}}(i) + E_t^A$ and $\omega_t(i) = W_t(i)/M_t$, and let us

define $\tau_{\text{id}} = \inf\{t \in [T] : \max_{i \in [K]} \omega_s(i) \leq \Delta/4, \quad \forall s \in \{t, \dots, T\}\}$, with $\inf \emptyset = T + 1$. Intuitively, it is the burn-in time for the uncertainty radius shrinks below the uniform gap Δ so that Algorithm 1 can identify this optimal arm. Finally, let us set τ as $\tau = \max\{\tau_{\text{stab}}, \tau_{\text{id}}\}$, after which we find the persistently optimal arm. The regret bound reads as follows.

Theorem 3. *Suppose Assumption 1 holds. Run Algorithm 1 with confidence parameter $\delta/2$. Then, with probability at least $1 - \delta$, $a_t = i^\dagger$, for any $t \geq \tau$. On the same event, $\bar{R}_T \leq \tau - 1$ and $R_T \leq \sum_{t=1}^{\tau-1} M_t$. We denote $\tau = T + 1$ when $\tau > T$.*

Once $t \geq \tau$, regret stops to grow. The rate of course depends on agent patterns M_t , pre-training errors (E_t^A), and the burn-in time τ , and is thus instance-dependent. As an example, if $M_t = \Theta(t)$, $|\mathcal{A}_t| \leq \bar{A}$, and $\bar{P}_m \leq Cpt^{-\alpha}$ with $\alpha > 0$, then $\tau \leq \tau_{\text{cf}} = O(\log T)$ under Assumption 2. If the remaining problem parameters are independent of T , $\bar{R}_T = O(\log T)$ and $R_T = O(M_0 \log T + \bar{A} \log^2 T)$. Thus, they recover the rate in closed systems [Landgren et al., 2016a] and in homogeneous open systems with Poisson processes [Xu and Klabjan, 2026], implying consistency while being applicable to more general regimes. Details on τ are in Appendix A.3.

4.2 Manifold-Based Learning

We first characterize the regret of Algorithm 2 for a general kernel, and then specialize the result to an intrinsic Matérn kernel to expose the dependence on the intrinsic dimension. Foremost, our construction of the estimate and confidence radius is valid w.r.t. $V_t(i)$.

Lemma 4. *Under the assumptions of Section 3.2, w.p. at least $1 - \delta$, for every $t \in [T]$ and $i \in [K]$, $|\hat{V}_t(i) - V_t(i)| \leq \beta_{i,t-1} \sigma_{i,t}^{\text{pop}}$.*

Then, controlling regret means controlling the radius. For arm i , let us define its information gain by $\Gamma_{i,t} = \frac{1}{2} \log \det(I + K_{i,t})$ where $K_{i,t}$ is the aforementioned Gram matrix. Let $\Delta\Gamma_t = \Gamma_{a_t,t} - \Gamma_{a_t,t-1}$ denote the information gained at round t . Importantly, we have $(\sigma_{a_t,t}^{\text{pop}})^2 \leq 2M_t^2 / \log(1 + M_t) \Delta\Gamma_t$. Let us define $\Gamma_T^\Sigma = \sum_{t=1}^T \Delta\Gamma_t = \sum_{i=1}^K \Gamma_{i,T}$, $N_T = \sum_{t=1}^T M_t$, and $H_T = \sum_{t=1}^T M_t^2 / \log(1 + M_t)$. Hence, H_T records the total amplification of uncertainty along the population path, Γ_T^Σ records the cumulative information gained to resolve uncertainty, and N_T is the total sample count.

Theorem 5. *Under the assumptions of Section 3.2, w.p. at least $1 - \delta$, $R_T \leq 2\beta_T \sqrt{2\Gamma_T^\Sigma H_T}$ with $\beta_T = S + R\sqrt{2(\bar{\gamma}_{N_T} + 1 + \log K/\delta)}$. By $\Gamma_T^\Sigma \leq \sum_{i=1}^K \bar{\gamma}_{N_i(T)} \leq K\bar{\gamma}_{N_T}$, we have $R_T \leq 2\beta_T \sqrt{2K\bar{\gamma}_{N_T} H_T}$.*

Theorem 5 exhibits a clean interaction between reward heterogeneity and population dynamics. Particularly, the term Γ_T^Σ captures the effective complexity of the reward landscape beyond the number of agents or the ambient dimension. The square root rate $\sqrt{\Gamma_T^\Sigma H_T}$ follows from summing the confidence widths over time and applying

Cauchy–Schwarz. Notably, there is another $\bar{\gamma}_{N_T}$ dependency through β_T , and thus the total rate is of $\sqrt{H_T \bar{\gamma}_{N_T}}$ consistent with the existing analysis machinery in RKHS. Regret is small when either the reward landscape is easy to learn, or the population trajectory limits the exposure of the uncertainty.

With an intrinsic Matérn kernel that exhibits a well understood spectrum, the regret dependence on the intrinsic dimension and smoothness becomes explicit. Assume that the agent representation space $\mathcal{X}$ is a smooth, compact, connected d -dimensional manifold without boundary, equipped with a Riemannian metric g . Let Δ_g denote the Laplace–Beltrami operator [Rosenberg, 1997], the intrinsic analogue of the Euclidean Laplacian on the manifold, and let $\{(\lambda_j, \varphi_j)\}_{j=0}^\infty$ denote its eigenpairs: $-\Delta_g \varphi_j = \lambda_j \varphi_j$ with $0 = \lambda_0 \leq \lambda_1 \leq \dots$. Here, the eigenfunctions φ_j describe spatial modes over the manifold, while the eigenvalues λ_j characterize their spatial frequencies. For $\nu > 0$ and $\ell > 0$, we define $a_\ell = 2\nu/\ell^2$ and consider the normalized intrinsic Matérn kernel $k_{\nu,\ell}(x, x') = 1/\kappa_{\nu,\ell} \sum_{j=0}^\infty (a_\ell + \lambda_j)^{-\nu-d/2} \varphi_j(x) \varphi_j(x')$, where constant $\kappa_{\nu,\ell} = \sup_x \sum_{j=0}^\infty (a_\ell + \lambda_j)^{-\nu-d/2} \varphi_j(x) \varphi_j(x)$ s.t. $k_{\nu,\ell}(x, x) \leq 1$ for every $x \in \mathcal{X}$. We then have:

Theorem 6. *For the Matérn kernel, w.p. at least $1 - \delta$,*

$$R_T \leq 2\bar{\beta}_T \sqrt{2C_\gamma K^{\frac{2\nu}{2\nu+d}} N_T^{\frac{d}{2\nu+d}} L_T^{\frac{2\nu}{2\nu+d}} H_T},$$

where $\bar{\beta}_T = S + R \sqrt{2[C_\gamma N_T^{\frac{d}{2\nu+d}} L_T^{\frac{2\nu}{2\nu+d}} + 1 + \log \frac{K}{\delta}]}$ and $L_T = \log(e + N_T)$.

Moreover, under bounded population size $M_t \leq \bar{M}$ for all t , we have $N_T \leq \bar{M}T$ and $H_T \leq T\bar{M}^2/\log(1+\bar{M})$. Hence, up to logarithmic factors and constants w.r.t. $(\mathcal{X}, g)$, ν , ℓ , S , R , and $\bar{M}$, $R_T = \tilde{O}(K^{\frac{\nu}{2\nu+d}} T^{\frac{1}{2} + \frac{d}{2\nu+d}})$. In particular, it is sublinear in T if $\nu > d/2$.

4.2.1 Parallel Exploration

Let us now echo back to the method where we allow agents to diversify arm selection. Recall that e_t is the number of agents who explore $\mathcal{C}_t$; $e_t = 0$ in the homogeneous exploration case. This distinction yields:

Theorem 7. *Under the assumptions of Section 3.2.1, with probability at least $1 - \delta$, $R_T \leq 2\beta_T \sqrt{2\Gamma_T^\Sigma H_T^{\text{het}}} + (2\beta_T + 1)N_T^{\text{exp}}$, where $N_T^{\text{exp}} = \sum_{t=1}^T e_t$ and $H_T^{\text{het}} = \sum_{t=1}^T \frac{(M_t - e_t)^4}{M_t^2 \log(1 + M_t - e_t)}$. If, in addition, parallel exploration is restricted to assignments (m, i) whose one-step information gain $I_{i,t}(m) = 1/2 \log(1 + \langle \phi(x_m), \mathcal{A}_{i,t-1}^{-1} \phi(x_m) \rangle_{\mathcal{H}_k})$ is at least $\eta > 0$, then $N_T^{\text{exp}} \leq \Gamma_T^\Sigma / \eta$, and $R_T \leq 2\beta_T \sqrt{2\Gamma_T^\Sigma H_T^{\text{het}}} + \frac{2\beta_T + 1}{\eta} \Gamma_T^\Sigma$. Lastly, $\Gamma_T^\Sigma \leq K\bar{\gamma}_{N_T}$ which allows us to plug in $\bar{\gamma}_{N_T}$.*

By the second argument in Theorem 7, the rate of parallel exploration depends on the matching mechanism for the budget e_t , which is our next focus. For $e_t > 0$, let

$\eta_t^* = \max_{\mathcal{P} \in \mathcal{MT}_t(e_t)} \min_{(m,i) \in \mathcal{P}} I_{i,t}(m)$ denote the largest one-step information threshold for which there exists a feasible matching whose every assigned pair attains at least that threshold. Indeed, $\mathcal{A}_{i,t-1} \succ 0$ and $\|\phi(x_m)\|_{\mathcal{H}_k}^2 = k(x_m, x_m) > 0$ imply $\langle \phi(x_m), \mathcal{A}_{i,t-1}^{-1} \phi(x_m) \rangle_{\mathcal{H}_k} > 0$, and hence $I_{i,t}(m) > 0$ for every feasible agent–arm pair. Since the set of feasible matchings $\mathcal{MT}_t(e_t)$ is finite and nonempty, it follows that $\eta_t^* > 0$. For $0 < \eta \leq \eta_t^*$, define the level set

$$\mathcal{MT}_t^\eta(e_t) = \{\mathcal{P} \in \mathcal{MT}_t(e_t) : I_{i,t}(m) \geq \eta \text{ for every } (m,i) \in \mathcal{P}\}$$

. We then choose $\mathcal{P}_t^{*,\eta} \in \arg \max_{\mathcal{P} \in \mathcal{MT}_t^\eta(e_t)} W_t(\mathcal{P})$. Again since $\mathcal{MT}_t(e_t)$ is finite, there exists $\tilde{\mathcal{P}}_t \in \mathcal{MT}_t(e_t)$ s.t. $\min_{(m,i) \in \tilde{\mathcal{P}}_t} I_{i,t}(m) = \eta_t^*$. Therefore, for every $0 < \eta \leq \eta_t^*$, $\tilde{\mathcal{P}}_t \in \mathcal{MT}_t^\eta(e_t)$, and hence $\mathcal{MT}_t^\eta(e_t) \neq \emptyset$. That said, $\mathcal{P}_t^{*,\eta}$ is well defined and meets the one-step information threshold.

We note that this matching itself may be suboptimal compared with the original optimal matching $\arg \max_{\mathcal{P} \in \mathcal{MT}_t(e_t)} W_t(\mathcal{P})$ over all possible matchings, due to the information-gain constraint. This further shows the intrinsic entanglement between uncertainty reduction (w.r.t. decision-centric quantity) and information gain (w.r.t. agent-level exploration). However, it is indeed possible for the two matchings to coincide, i.e., to achieve the best of both worlds. To see this, by Cauchy-Schwarz, $G_{i,t}(m) \leq (\sigma_{i,t}^{\text{pop}})^2 (1 - e^{-2I_{i,t}(m)})$ (2), i.e., $G_{i,t}(m)/(\sigma_{i,t}^{\text{pop}})^2 \geq 1 - e^{-2\eta} \implies I_{i,t}(m) \geq \eta$. Thus, if the original optimal matching meets the condition on $G_{i,t}(m)$, then it belongs to $\mathcal{MT}_t^\eta(e_t)$ and is also the maximizer of $W_t(\mathcal{P})$ therein.

Homogeneous versus Parallel If no exploratory assignments are made, so that $e_t = 0$ for every t , then $M_t - e_t = M_t$, $D_T = N_T^{\text{exp}} = 0$, and $H_T^{\text{het}} = \sum_{t=1}^T \frac{M_t^2}{\log(1+M_t)} = H_T$. Theorem 7 therefore reduces to the regret bound of Theorem 5, implying consistency. Generally, if $e_t > 0$, the regret bound achieves a reduction in the first term by replacing H_T with H_T^{het} , but at the price of the extra exploration reflected in the second term. The bound for parallel exploration is smaller whenever $2\beta_T \sqrt{2(K\bar{\gamma}_{N_T})}(\sqrt{H_T} - \sqrt{H_T^{\text{het}}}) > (2\beta_T + 1)/\eta(K\bar{\gamma}_{N_T})$. Since η_t^* is the largest threshold over all feasible matchings of size e_t , it is equivalent to

$$\min_{t: e_t > 0} \eta_t^* > (2\beta_T + 1)/2\sqrt{2\beta_T} \sqrt{K\bar{\gamma}_{N_T}}/(\sqrt{H_T} - \sqrt{H_T^{\text{het}}}),$$

provided $H_T^{\text{het}} < H_T$. Increasing e_t reduces H_T^{het} , reflecting more parallel exploration and favorably lowering the threshold on η_t^* . However, it may also decrease η_t^* . This presents a nontrivial trade-off, particularly without the hindsight information about $\min_{t: e_t > 0} \eta_t^*$.

4.3 Computational Complexity

While we see a progressive improvement in regret, except for the trade-offs between homogeneous and heterogeneous exploration, their computational complexity might also be a valid factor to take into account at deployment time. We observe a progressive increase in this complexity from agent-wise learning to manifold-based learning, while the two manifold-based methods are more or less equivalent. While less desirable, this is perhaps the price we pay for the fact that *“all past observations remain useful even after agents have departed given the manifold.”* We hope to convey this.

Specifically, Agent-wise Global-UCB has lower computational overhead. At round t , it takes $O(KM_t)$ to compute the estimates and confidence radii of all K arms over the M_t agents. The manifold methods inherit the computational cost of kernel regression. Recall that $N_i(t-1)$ and N_{t-1} are the numbers of observations of arm i and all arms, respectively, by $t-1$; thus $N_{t-1} = \sum_{s=1}^{t-1} M_s$. Evaluating the UCBs of all arms takes $O(N_{t-1}M_t + M_t^2 + \sum_{i=1}^K N_i(t-1)^2)$ time. Here, the first two terms arise from constructing the kernel quantities $q_{i,t}$ and c_t , while the last term is the cost of computing $(K_{i,t-1} + I)^{-1}q_{i,t}$ when evaluating $\sigma_{i,t}^{\text{pop}}$. After observing the batch of M_t agents, the kernel matrix of the selected arm grows from dimension $N_{a_t}(t-1)$ to $N_{a_t}(t-1) + M_t$. Reusing the inverse of the kernel matrix from the previous round, incorporating one new observation requires updating the inverse after adding one new row and column. This update involves $O(n^2)$ entries when the kernel matrix has size $n \times n$. Summing this cost over the M_t new observations gives $O(M_t(N_{a_t}(t-1) + M_t)^2)$. Thus, the overall per-round complexity is $O(M_t(N_{t-1} + M_t)^2) = O(M_t(\sum_{s=1}^t M_s)^2)$ in the worst case. Parallel exploration incurs extra cost of evaluating the assignments and solving the matching problem; however, none dominates $O(M_t(\sum_{s=1}^t M_s)^2)$ in the worst case. More details are in Appendix A.4.

5 Conclusions

Learning in open systems is intricate under reward heterogeneity and population dynamics, and the findings herein show different difficulties of openness. In the general setting, new agents contribute uncertainty through their pre-training errors, so regret depends on both these errors and their lifetimes in the system, with these dependencies being individually minimax tight up to logarithmic factors. When a stable optimal arm exists, regret is instead governed by the time needed to identify the persistent optimum. Both depend on population dynamics. Through the manifold, observations transfer across agents via their common geometry, replacing separate control of agent-wise uncertainty with direct control of uncertainty in the aggregate reward. The regret then depends on the cumulative information gain and the population path, coupling the complexity of learning with the population dynamics that scale uncertainty.

AI Use Statement

Generative AI tools, particularly GPT-5.6 Sol, have assisted the authors with manuscript polishing, drafting the proofs of the mathematical claims, and reviewing the content. The tool does not participate in the ideation and conceptualization process of the paper. The authors have independently revised the content and take responsibility for the final content of this work, including text, claims, or artifacts produced with the aid of generative AI.

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A Appendix

A.1 Pseudo code

Algorithm 1: Agent-Wise Global-UCB

Input: Arms K , horizon T , confidence level δ

Output: Common arm choices $\{a_t\}_{t \in [T]}$

```

1 for  $t = 1, 2, \dots, T$  do
2   Observe  $\mathcal{M}_t$ , and set  $\mathcal{A}_t = \mathcal{M}_t \setminus \mathcal{M}_{t-1}$  and  $\mathcal{R}_t = \mathcal{M}_t \cap \mathcal{M}_{t-1}$ 
3   foreach  $m \in \mathcal{A}_t$  do
4     Assign its arrival index  $j(m)$ , obtain  $\{\hat{\mu}_{m,i}^{\text{entry}}\}_{i \in [K]}$  and  $\bar{P}_m$ , and set
        $n_{m,i}(t-1) = 0$ 
5   foreach  $m \in \mathcal{R}_t, i \in [K]$  do
6     Set  $\rho_{m,i}(t-1) \leftarrow \min\{\bar{P}_m, c_m(n_{m,i}(t-1))\}$ , and set  $\hat{\mu}_{m,i}(t-1) \leftarrow \hat{\mu}_{m,i}^{\text{entry}}$  if
        $\bar{P}_m \leq c_m(n_{m,i}(t-1))$ , and  $\hat{\mu}_{m,i}(t-1) \leftarrow \bar{r}_{m,i}(t-1)$  otherwise
7   foreach  $i \in [K]$  do
8     Set  $\hat{V}_t(i) \leftarrow \sum_{m \in \mathcal{R}_t} \hat{\mu}_{m,i}(t-1) + \sum_{m \in \mathcal{A}_t} \hat{\mu}_{m,i}^{\text{entry}}$  and
        $B_t^{\text{stat}}(i) \leftarrow \sum_{m \in \mathcal{R}_t} \rho_{m,i}(t-1)$ 
9   Select  $a_t \in \arg \max_{i \in [K]} \{\hat{V}_t(i) + B_t^{\text{stat}}(i)\}$ 
10  foreach  $m \in \mathcal{M}_t$  do
11    Pull  $a_t$ , observe  $r_m^{a_t}(t)$ , and update  $n_{m,a_t}(t)$  and  $\bar{r}_{m,a_t}(t)$ 

```

A.2 Special cases for Theorem 1

Model	Entry estimate	Valid entry radius $\bar{P}_m$
Linear	$x_m^\top \hat{\theta}_i$	$\max_{i \in [K]} x_m^\top (\hat{\theta}_i - \theta_i) $
Nonlinear parametric	$f_i(x_m, \hat{\theta}_i)$	$\max_{i \in [K]} L_i \ \hat{\theta}_i - \theta_i\ _2$
Clustered	$\hat{\theta}_{c(m)}^i$	$\max_{i \in [K]} \hat{\theta}_{c(m)}^i - \theta_{c(m)}^i $
Zero knowledge	0	1

Table 1: Examples of entry information for newly arriving agents.

In the linear and nonlinear cases, estimates of shared parameters can be transferred to a newly arriving agent through its observed features; in the clustered case, the agent inherits information associated with its cluster; and zero-knowledge initialization provides the worst-case valid radius under rewards in $[0, 1]$. Once $\bar{P}_m$ is specified, all cases are covered by the same regret bound in Theorem 1.

Algorithm 2: Manifold-based Learning

Input: Kernel k , RKHS bound S , noise parameter R , confidence level δ ,
information-gain bound $\{\bar{\gamma}_n\}_{n \geq 0}$

Output: Common arm choices $\{a_t\}_{t \in [T]}$

```
1 Initialize  $\mathcal{D}_i \leftarrow \emptyset$  and  $N_i(0) \leftarrow 0$  for every  $i \in [K]$ 
2 for  $t = 1, 2, \dots, T$  do
3   Observe  $\mathcal{M}_t$  and  $\{x_m : m \in \mathcal{M}_t\}$ , and set  $c_t \leftarrow \sum_{m,n \in \mathcal{M}_t} k(x_m, x_n)$ 
4   foreach  $i \in [K]$  do
5     Enumerate  $\mathcal{D}_i = \{(x_{i,\ell}, y_{i,\ell})\}_{\ell=1}^{N_i(t-1)}$ , and form the Gram matrix
6      $[K_{i,t-1}]_{\ell\ell'} \leftarrow k(x_{i,\ell}, x_{i,\ell'})$ 
7     Set  $[q_{i,t}]_\ell \leftarrow \sum_{m \in \mathcal{M}_t} k(x_{i,\ell}, x_m)$ 
8     Set  $\hat{V}_t(i) \leftarrow q_{i,t}^\top (K_{i,t-1} + I)^{-1} y_{i,t-1}$  and  $\sigma_{i,t}^{\text{pop}} \leftarrow \sqrt{c_t - q_{i,t}^\top (K_{i,t-1} + I)^{-1} q_{i,t}}$ 
9     Set  $\beta_{i,t-1} \leftarrow S + R\sqrt{2(\bar{\gamma}_{N_i(t-1)} + 1 + \log(K/\delta))}$ 
10    Select  $a_t \in \arg \max_{i \in [K]} \{\hat{V}_t(i) + \beta_{i,t-1} \sigma_{i,t}^{\text{pop}}\}$ 
11    Every  $m \in \mathcal{M}_t$  pulls  $a_t$  and observes  $r_m^{a_t}(t)$ 
12    Update  $\mathcal{D}_{a_t} \leftarrow \mathcal{D}_{a_t} \cup \{(x_m, r_m^{a_t}(t)) : m \in \mathcal{M}_t\}$  and  $N_{a_t}(t) \leftarrow N_{a_t}(t-1) + M_t$ 
```

A.3 Characterization of τ

Recall that

$$\tau = \max\{\tau_{\text{stab}}, \tau_{\text{id}}\},$$

where τ_{stab} is the time after which the persistent global optimum in Assumption 1 is established, and τ_{id} is the time after which the uncertainty in the Global-UCB indices is too small to overturn this ordering. In general, τ_{stab} is instance-dependent. We first give one sufficient condition under which it admits an explicit bound and then characterize τ_{id} .

One sufficient condition for population stability. The persistent-optimum condition does not require any particular distribution of the agent reward vectors. The following stationary population model provides one sufficient condition under which it holds with high probability.

Assumption 2 (Stationary population composition). *There are no departures, and the reward-mean vectors $\mu_m = (\mu_m^1, \dots, \mu_m^K) \in [0, 1]^K$ of all initial and arriving agents are drawn independently from a common distribution, independently of the arrival process. Let $v(i) = \mathbb{E}[\mu_m^i]$, and suppose that v has a unique maximizer $i^\dagger \in [K]$, with gap $\Delta_0 = v(i^\dagger) - \max_{i \neq i^\dagger} v(i) > 0$.*

Fix $0 < \Delta < \Delta_0$, and define $N_{\text{stab}} = \frac{2}{(\Delta_0 - \Delta)^2} \log \frac{4KT}{\delta}$. Let

$$\tau_{\text{pop}} = \inf \{t \in [T] : M_s \geq N_{\text{stab}}, \quad \forall s \in \{t, \dots, T\}\},$$

with $\inf \emptyset = T+1$. Under Assumption 2, there are no departures, so M_t is nondecreasing and

$$\tau_{\text{pop}} = \inf \{t \in [T] : M_t \geq N_{\text{stab}}\}.$$

Lemma 8 (Population stability). *Under Assumption 2, with probability at least $1 - \delta/2$,*

$$\bar{V}_t(i^\dagger) - \max_{i \neq i^\dagger} \bar{V}_t(i) \geq \Delta, \quad \forall t \geq \tau_{\text{pop}}, \quad t \leq T.$$

Consequently, on this event, if $\tau_{\text{pop}} \leq T$, Assumption 1 holds with $\tau_{\text{stab}} \leq \tau_{\text{pop}}$.

Proof. For a fixed t and i , conditional on M_t , $\bar{V}_t(i)$ is the average of M_t independent random variables in $[0, 1]$ with common mean $v(i)$. Therefore, Hoeffding's inequality gives

$$\Pr\left(\left|\bar{V}_t(i) - v(i)\right| > \frac{\Delta_0 - \Delta}{2} \mid M_t\right) \leq 2 \exp\left(-\frac{M_t(\Delta_0 - \Delta)^2}{2}\right).$$

Whenever $M_t \geq N_{\text{stab}}$, the right-hand side is at most $\delta/(2KT)$. A union bound over all $t \in [T]$ and $i \in [K]$ therefore gives, with probability at least $1 - \delta/2$,

$$\max_{i \in [K]} \left|\bar{V}_t(i) - v(i)\right| \leq \frac{\Delta_0 - \Delta}{2}$$

at every round satisfying $M_t \geq N_{\text{stab}}$.

Hence, for every such round and every $i \neq i^\dagger$,

$$\begin{aligned} \bar{V}_t(i^\dagger) - \bar{V}_t(i) &\geq v(i^\dagger) - v(i) - \left|\bar{V}_t(i^\dagger) - v(i^\dagger)\right| - \left|\bar{V}_t(i) - v(i)\right| \\ &\geq \Delta_0 - (\Delta_0 - \Delta) = \Delta. \end{aligned}$$

If $\tau_{\text{pop}} \leq T$, then by the definition of τ_{pop} , $M_t \geq N_{\text{stab}}$ for every $t \geq \tau_{\text{pop}}$. Thus the persistent-optimum condition in Assumption 1 holds from τ_{pop} onward, and we may take $\tau_{\text{stab}} \leq \tau_{\text{pop}}$. $\square$

Global-UCB confidence. Define

$$\mathcal{E}_{\text{UCB}} = \left\{ \left| \hat{V}_t(i) - V_t(i) \right| \leq B_t^{\text{stat}}(i) + E_t^A, \quad \forall t \in [T], \quad i \in [K] \right\}.$$

Lemma 9 (Simultaneous confidence for Global-UCB). *Suppose rewards lie in $[0, 1]$ and the entry estimates satisfy $\max_{i \in [K]} \left| \hat{\mu}_{m,i}^{\text{entry}} - \mu_m^i \right| \leq \bar{P}_m$ for every agent m . If Algorithm 1 is run with confidence parameter $\delta/2$, then*

$$\mathbb{P}(\mathcal{E}_{\text{UCB}}) \geq 1 - \frac{\delta}{2}.$$

Proof. When Algorithm 1 is run with confidence parameter $\delta/2$, its anytime radius is

$$c_m(n) = \sqrt{\frac{1}{2n} \log \left(\frac{4Kj(m)(j(m)+1)n(n+1)}{\delta} \right)}, \quad n \geq 1.$$

For the $j(m)$ -th agent, arm i , and sample size $n \geq 1$, Hoeffding's inequality gives

$$\begin{aligned} \mathbb{P}(|\bar{r}_{m,i}(n) - \mu_m^i| > c_m(n)) &\leq 2 \exp\{-2nc_m(n)^2\} \\ &= \frac{\delta}{2Kj(m)(j(m)+1)n(n+1)}. \end{aligned}$$

Since

$$\sum_{j=1}^{\infty} \frac{1}{j(j+1)} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1,$$

a union bound over all agents, arms, and sample sizes yields

$$\mathbb{P}(|\bar{r}_{m,i}(n) - \mu_m^i| \leq c_m(n), \quad \forall m, i, n \geq 1) \geq 1 - \frac{\delta}{2}.$$

On this event, for every continuing agent $m \in \mathcal{R}_t$,

$$|\hat{\mu}_{m,i}(t-1) - \mu_m^i| \leq \rho_{m,i}(t-1) = \min\{\bar{P}_m, c_m(n_{m,i}(t-1))\},$$

whereas for every arriving agent $m \in \mathcal{A}_t$,

$$|\hat{\mu}_{m,i}^{\text{entry}} - \mu_m^i| \leq \bar{P}_m.$$

Therefore,

$$\begin{aligned} |\hat{V}_t(i) - V_t(i)| &\leq \sum_{m \in \mathcal{R}_t} \rho_{m,i}(t-1) + \sum_{m \in \mathcal{A}_t} \bar{P}_m \\ &= B_t^{\text{stat}}(i) + E_t^A \end{aligned}$$

simultaneously for every $t \in [T]$ and $i \in [K]$. Hence

$$\mathbb{P}(\mathcal{E}_{\text{UCB}}) \geq 1 - \frac{\delta}{2}.$$

□

Identification by Global-UCB. Recall

$$W_t(i) = B_t^{\text{stat}}(i) + E_t^A$$

and define

$$\omega_t(i) = \frac{W_t(i)}{M_t}.$$

The identification time is

$$\tau_{\text{id}} = \inf \left\{ t \in [T] : \max_{i \in [K]} \omega_s(i) \leq \frac{\Delta}{4}, \quad \forall s \in \{t, \dots, T\} \right\},$$

with $\inf \emptyset = T + 1$. Thus, once $t \geq \tau_{\text{id}}$,

$$B_t^{\text{stat}}(i) + E_t^A \leq \frac{M_t \Delta}{4}, \quad \forall i \in [K].$$

This definition is pathwise and does not require an additional probabilistic event.

We next give a sufficient upper bound on τ_{id} .

Proposition 1 (Closed-form upper bound on τ_{id}). *Suppose there are no departures, $M_t \geq \underline{\lambda}t$, $|\mathcal{A}_t| \leq \bar{A}$, and $\bar{P}_m \leq C_P t^{-\alpha}$ for every $t \in [T]$ and $m \in \mathcal{A}_t$, where $\underline{\lambda}, \bar{A}, C_P > 0$ and $\alpha \geq 0$. Let $P_0 = \sum_{m \in \mathcal{M}_0} \bar{P}_m$.*

If $\alpha = 0$ and $\bar{A}C_P/\underline{\lambda} < \Delta/4$, then $\tau_{\text{id}} \leq \max\{1, \lceil P_0/[\underline{\lambda}(\Delta/4 - \bar{A}C_P/\underline{\lambda})] \rceil\}$.

If $0 < \alpha < 1$, then $\tau_{\text{id}} \leq \left\lceil \max\left\{8P_0/(\underline{\lambda}\Delta), [8\bar{A}C_P/(\underline{\lambda}(1-\alpha)\Delta)]^{1/\alpha}\right\} \right\rceil$.

If $\alpha = 1$, let $A_\Delta = \max\{1, 8\bar{A}C_P/(\underline{\lambda}\Delta)\}$. Then $\tau_{\text{id}} \leq \lceil \max\{8P_0/(\underline{\lambda}\Delta), 2A_\Delta \log(2eA_\Delta)\} \rceil$.

If $\alpha > 1$, then $\tau_{\text{id}} \leq \left\lceil 4(P_0 + \frac{\alpha}{\alpha-1}\bar{A}C_P)/(\underline{\lambda}\Delta) \right\rceil$.

Proof. Since $\rho_{m,i}(t-1) \leq \bar{P}_m$, for every arm i ,

$$W_t(i) \leq \sum_{m \in \mathcal{M}_t} \bar{P}_m \leq P_0 + \bar{A}C_P H_\alpha(t), \quad H_\alpha(t) = \sum_{s=1}^t s^{-\alpha}.$$

Hence, using $M_t \geq \underline{\lambda}t$,

$$\max_{i \in [K]} \omega_t(i) \leq \frac{P_0 + \bar{A}C_P H_\alpha(t)}{\underline{\lambda}t}. \quad (\text{A})$$

If $\alpha = 0$, then $H_0(t) = t$, so

$$\max_i \omega_t(i) \leq \frac{P_0}{\underline{\lambda}t} + \frac{\bar{A}C_P}{\underline{\lambda}}.$$

Under $\bar{A}C_P/\underline{\lambda} < \Delta/4$, the right-hand side is at most $\Delta/4$ whenever

$$t \geq \frac{P_0}{\underline{\lambda}(\Delta/4 - \bar{A}C_P/\underline{\lambda})}.$$

It is nonincreasing in t , yielding the first claim; the additional maximum with 1 covers $P_0 = 0$.

If $0 < \alpha < 1$, then $H_\alpha(t) \leq t^{1-\alpha}/(1-\alpha)$. Thus, by (A),

$$\max_i \omega_t(i) \leq \frac{P_0}{\underline{\lambda}t} + \frac{\bar{A}C_P}{\underline{\lambda}(1-\alpha)} t^{-\alpha}.$$

The stated threshold makes each term at most $\Delta/8$, and both terms are nonincreasing in t , proving the second claim.

If $\alpha = 1$, then $H_1(t) \leq \log(et)$, and hence

$$\max_i \omega_t(i) \leq \frac{P_0}{\underline{\lambda}t} + \frac{\bar{A}C_P}{\underline{\lambda}} \frac{\log(et)}{t}.$$

The first term is at most $\Delta/8$ when $t \geq 8P_0/(\underline{\lambda}\Delta)$. Moreover, $\bar{A}C_P/\underline{\lambda} \leq A_\Delta\Delta/8$. Let $t_0 = 2A_\Delta \log(2eA_\Delta)$. Since $B = \log(2eA_\Delta) > 1$,

$$A_\Delta \log(et_0) = A_\Delta(B + \log B) \leq 2A_\Delta B = t_0.$$

Because $\log(et)/t$ is nonincreasing for $t \geq 1$, the same inequality holds for every $t \geq t_0$. Thus the second term is also at most $\Delta/8$, proving the third claim.

Finally, if $\alpha > 1$, then

$$H_\alpha(t) \leq 1 + \int_1^\infty x^{-\alpha} dx = \frac{\alpha}{\alpha-1}.$$

Therefore,

$$\max_i \omega_t(i) \leq \frac{P_0 + \frac{\alpha}{\alpha-1} \bar{A}C_P}{\underline{\lambda}t},$$

which is at most $\Delta/4$ under the stated threshold and remains so thereafter. This proves the final claim. $\square$

Closed-form burn-in bound. We now combine the two components under the stationary population model. Under Assumption 2 and the conditions of Proposition 1,

$$\tau_{\text{pop}} \leq \left\lceil \frac{N_{\text{stab}}}{\underline{\lambda}} \right\rceil.$$

Let

$$\tau_{\text{id}}^{\text{ub}}$$

denote the corresponding upper bound in Proposition 1, and define

$$\tau_{\text{cf}} = \max \left\{ \left\lceil \frac{N_{\text{stab}}}{\underline{\lambda}} \right\rceil, \tau_{\text{id}}^{\text{ub}} \right\}.$$

If $\tau_{\text{cf}} \leq T$, then $\tau_{\text{pop}} \leq T$, and Lemma 8 implies that, with probability at least $1 - \delta/2$,

$$\tau_{\text{stab}} \leq \tau_{\text{pop}} \leq \left\lceil \frac{N_{\text{stab}}}{\underline{\lambda}} \right\rceil.$$

Therefore, on the same event,

$$\tau = \max\{\tau_{\text{stab}}, \tau_{\text{id}}\} \leq \tau_{\text{cf}}.$$

If $\tau_{\text{cf}} > T$, then $\min\{T, \tau_{\text{cf}}\} = T$, and the corresponding finite-horizon regret bounds below follow directly from the trivial bounds over the first T rounds.

Corollary 9.1 (Explicit stable-regime regret bounds). *Suppose Assumption 2 and the conditions of Proposition 1 hold, and run Algorithm 1 with confidence parameter $\delta/2$. Then, with probability at least $1 - \delta$,*

$$\bar{R}_T \leq \min\{T, \tau_{\text{cf}}\}.$$

If, in addition, $M_t \leq M_0 + \bar{A}t$, $\forall t \in [T]$, and $q_T = \min\{T, \tau_{\text{cf}}\}$, then

$$R_T \leq M_0 q_T + \frac{\bar{A} q_T (q_T + 1)}{2}.$$

Proof. Let

$$q_T = \min\{T, \tau_{\text{cf}}\}.$$

We consider two cases. First, suppose $\tau_{\text{cf}} \leq T$. Since

$$\tau_{\text{pop}} \leq \left\lceil \frac{N_{\text{stab}}}{\underline{\lambda}} \right\rceil \leq \tau_{\text{cf}} \leq T,$$

Lemma 8 implies, with probability at least $1 - \delta/2$, that Assumption 1 holds with

$$\tau_{\text{stab}} \leq \tau_{\text{pop}} \leq \left\lceil \frac{N_{\text{stab}}}{\underline{\lambda}} \right\rceil.$$

By Lemma 9, the simultaneous Global-UCB confidence event holds with probability at least $1 - \delta/2$. Hence, by a union bound, both events hold simultaneously with probability at least $1 - \delta$. On this event,

$$\tau = \max\{\tau_{\text{stab}}, \tau_{\text{id}}\} \leq \tau_{\text{cf}}.$$

Theorem 3 therefore gives

$$\bar{R}_T \leq \tau_{\text{cf}} - 1 \leq q_T.$$

Similarly,

$$R_T \leq \sum_{t=1}^{\tau_{\text{cf}}-1} M_t \leq \sum_{t=1}^{q_T} M_t.$$

Now suppose $\tau_{\text{cf}} > T$. Then $q_T = T$. Since each population-average reward lies in $[0, 1]$,

$$\bar{R}_T \leq T = q_T,$$

and likewise

$$R_T \leq \sum_{t=1}^T M_t = \sum_{t=1}^{q_T} M_t.$$

Thus, in both cases,

$$\bar{R}_T \leq q_T = \min\{T, \tau_{\text{cf}}\}, \quad R_T \leq \sum_{t=1}^{q_T} M_t.$$

If, in addition, $M_t \leq M_0 + \bar{A}t$ for all $t \in [T]$, then

$$R_T \leq \sum_{t=1}^{q_T} (M_0 + \bar{A}t) = M_0 q_T + \frac{\bar{A} q_T (q_T + 1)}{2}.$$

□

Rate implications. The closed-form bound separates the time needed for population stability from the time needed for identification. Under the stationary population model,

$$\frac{N_{\text{stab}}}{\underline{\lambda}} = O\left(\frac{1}{\underline{\lambda}(\Delta_0 - \Delta)^2} \log \frac{KT}{\delta}\right).$$

In contrast, the bounds on τ_{id} in Proposition 1 do not grow with T when the remaining parameters are independent of T . For example, when $\bar{P}_m = O(t^{-1/2})$ for agents arriving at round t ,

$$\tau_{\text{id}} = O(\Delta^{-2}),$$

up to the population-growth and entry-error constants, while $\bar{P}_m = O(t^{-1})$ yields

$$\tau_{\text{id}} = \tilde{O}(\Delta^{-1}).$$

Consequently, if $K, \delta, \underline{\lambda}, \Delta_0, \Delta, \bar{A}, C_P, \alpha$, and P_0 are independent of T , with

$$\frac{\bar{A}C_P}{\underline{\lambda}} < \frac{\Delta}{4}$$

when $\alpha = 0$, then

$$\tau_{\text{cf}} = O(\log T).$$

If also

$$M_t \leq M_0 + \bar{A}t,$$

then

$$\bar{R}_T = O(\log T), \quad R_T = O(M_0 \log T + \bar{A} \log^2 T).$$

Thus, under this sufficient population model, the population may continue to grow throughout the horizon, while regret accumulates only during a logarithmic burn-in period. More generally, Theorem 3 does not require the stationary population model: any instance satisfying Assumption 1 admits the same post- τ identification guarantee, with the resulting regret determined by its instance-dependent values of τ_{stab} and τ_{id} .

A.4 Computation Complexity

The manifold methods inherit the computational cost of exact kernel regression. Recall that $N_i(t-1)$ and N_{t-1} are the numbers of observations of arm i and all arms, respectively, by $t-1$. For each arm i , the vector $q_{i,t}$ has $N_i(t-1)$ entries, each involving M_t kernel evaluations. Constructing $q_{i,t}$ for all arms therefore takes $O(N_{t-1}M_t)$ time. Similarly, c_t contains M_t^2 pairwise kernel evaluations and takes $O(M_t^2)$ time.

To evaluate $\hat{V}_t(i)$ and $\sigma_{i,t}^{\text{pop}}$, we solve systems involving $K_{i,t-1} + I$. Reusing the matrix decomposition maintained from previous observations, each solve takes $O(N_i(t-1)^2)$ time, and hence the corresponding cost over all arms is $O(\sum_{i=1}^K N_i(t-1)^2)$. Altogether, evaluating the UCB indices takes $O(N_{t-1}M_t + M_t^2 + \sum_{i=1}^K N_i(t-1)^2)$ time.

After the common arm a_t is selected, M_t new observations are added to its kernel matrix. Updating the matrix decomposition once for each new observation costs at most $O((N_{a_t}(t-1) + M_t)^2)$ per observation, and therefore $O(M_t(N_{a_t}(t-1) + M_t)^2)$ for the batch. Combining these computations, the overall per-round complexity is $O(M_t(N_{t-1} + M_t)^2)$ in the worst case, which reduces to $O(N_{t-1}^2)$ when the population size is uniformly bounded.

Parallel exploration incurs the same cost from the UCB-style evaluation, together with the costs of evaluating the assignment gains and solving the matching problem. For a candidate pair (m, i) , directly evaluating $G_{i,t}(m)$ requires solving a kernel system of dimension $N_i(t-1)$, which takes $O(N_i(t-1)^2)$ time using the maintained matrix decomposition. There are M_t candidate agents for each $i \in \mathcal{C}_t$, giving $O(M_t \sum_{i \in \mathcal{C}_t} N_i(t-1)^2)$ time for all assignment gains.

The matching can be formulated as a minimum-cost flow problem on a bipartite network with $M_t + |\mathcal{C}_t| + 2$ vertices and $M_t|\mathcal{C}_t| + M_t + |\mathcal{C}_t|$ edges, where e_t units of flow correspond to the selected agent-arm pairs. Using a network-flow algorithm, the matching can be solved in $O(e_t M_t |\mathcal{C}_t| \log(M_t + |\mathcal{C}_t|))$ time, which is at most $O(M_t |\mathcal{C}_t| \min\{M_t, |\mathcal{C}_t|\} \log(M_t + |\mathcal{C}_t|))$.

Since $M_t \sum_{i \in \mathcal{C}_t} N_i(t-1)^2 \leq M_t N_{t-1}^2$ and $\sum_{i=1}^K M_{t,i} (N_i(t-1) + M_{t,i})^2 \leq M_t (N_{t-1} + M_t)^2$, the overall per-round complexity is $O(M_t(N_{t-1} + M_t)^2 + M_t |\mathcal{C}_t| \min\{M_t, |\mathcal{C}_t|\} \log(M_t + |\mathcal{C}_t|))$.

$|\mathcal{C}_t|)$). For fixed K , the dominating term is still $O(M_t(N_{t-1} + M_t)^2)$, which is consistent with the homogeneous exploration case. For a uniformly bounded population size the complexity again reduces to $O(N_{t-1}^2)$.

A.5 Proof of Results in General Systems

A.5.1 Proof of Theorem 1

Proof. Fix an agent m , an arm i , and a sample size $n \geq 1$. By Hoeffding's inequality and the definition

$$c_m(n) = \sqrt{\frac{1}{2n} \log \left(\frac{2Kj(m)(j(m) + 1)n(n + 1)}{\delta} \right)},$$

we have

$$\Pr(|\bar{r}_{m,i}(n) - \mu_m^i| > c_m(n)) \leq 2 \exp\{-2nc_m(n)^2\} = \frac{\delta}{Kj(m)(j(m) + 1)n(n + 1)}.$$

Since the agents are indexed by their order of first appearance and

$$\sum_{j=1}^{\infty} \frac{1}{j(j + 1)} = \sum_{n=1}^{\infty} \frac{1}{n(n + 1)} = 1,$$

a union bound over all agents, arms, and sample sizes gives, with probability at least $1 - \delta$,

$$|\bar{r}_{m,i}(n) - \mu_m^i| \leq c_m(n)$$

simultaneously for every agent m , arm i , and $n \geq 1$. We work on this event throughout the remainder of the proof.

Together with the entry guarantee

$$\max_{i \in [K]} |\hat{\mu}_{m,i}^{\text{entry}} - \mu_m^i| \leq \bar{P}_m,$$

this implies, for every continuing agent $m \in \mathcal{R}_t$,

$$|\hat{\mu}_{m,i}(t - 1) - \mu_m^i| \leq \rho_{m,i}(t - 1) = \min \{ \bar{P}_m, c_m(n_{m,i}(t - 1)) \}.$$

For every newly arriving agent $m \in \mathcal{A}_t$,

$$|\hat{\mu}_{m,i}^{\text{entry}} - \mu_m^i| \leq \bar{P}_m.$$

Therefore, for every arm i ,

$$\begin{aligned} |\hat{V}_t(i) - V_t(i)| &\leq \sum_{m \in \mathcal{R}_t} \rho_{m,i}(t - 1) + \sum_{m \in \mathcal{A}_t} \bar{P}_m \\ &= B_t^{\text{stat}}(i) + E_t^A. \end{aligned}$$

Let

$$i_t^* \in \arg \max_{i \in [K]} V_t(i).$$

Since E_t^A is common across arms, the UCB rule gives

$$\widehat{V}_t(i_t^*) + B_t^{\text{stat}}(i_t^*) \leq \widehat{V}_t(a_t) + B_t^{\text{stat}}(a_t).$$

Using the confidence bounds on both i_t^* and a_t ,

$$\begin{aligned} V_t(i_t^*) &\leq \widehat{V}_t(i_t^*) + B_t^{\text{stat}}(i_t^*) + E_t^A \\ &\leq \widehat{V}_t(a_t) + B_t^{\text{stat}}(a_t) + E_t^A \\ &\leq V_t(a_t) + 2B_t^{\text{stat}}(a_t) + 2E_t^A. \end{aligned}$$

Hence

$$V_t(i_t^*) - V_t(a_t) \leq 2B_t^{\text{stat}}(a_t) + 2E_t^A.$$

Because all active agents select the common arm a_t ,

$$R_T = \sum_{t=1}^T [V_t(i_t^*) - V_t(a_t)],$$

and therefore

$$R_T \leq 2 \sum_{t=1}^T B_t^{\text{stat}}(a_t) + 2 \sum_{t=1}^T E_t^A.$$

It remains to control the first term. By the definition of B_t^{stat} ,

$$\begin{aligned} \sum_{t=1}^T B_t^{\text{stat}}(a_t) &= \sum_{t=1}^T \sum_{m \in \mathcal{R}_t} \rho_{m,a_t}(t-1) \\ &= \sum_{m: L_m > 0} \sum_{i=1}^K \sum_{\substack{t \leq T: \\ m \in \mathcal{R}_t, a_t=i}} \rho_{m,i}(t-1). \end{aligned}$$

Thus, the outer sum is over distinct agents, while L_m records the total number of rounds in which agent m contributes to this term.

Fix an agent m with $L_m > 0$ and an arm i , and write

$$n = N_{m,i}^R(T).$$

Since $\rho_{m,i}(t-1) \leq \bar{P}_m$,

$$\sum_{\substack{t \leq T: \\ m \in \mathcal{R}_t, a_t=i}} \rho_{m,i}(t-1) \leq n \bar{P}_m.$$

We next derive an alternative statistical bound. Let

$$t_1 < t_2 < \dots < t_n$$

be the rounds at which $m \in \mathcal{R}_{t_r}$ and $a_{t_r} = i$. The first such round contributes at most $\bar{P}_m$. Before round t_r , $r \geq 2$, agent m has already observed arm i on at least $r - 1$ previous rounds. Moreover, $c_m(\cdot)$ is nonincreasing on the positive integers. Indeed, writing

$$C_m = \frac{2Kj(m)(j(m) + 1)}{\delta},$$

the function

$$x \mapsto \frac{\log(C_m x(x + 1))}{x}$$

is decreasing for $x \geq 1$, since $C_m \geq 4$ and

$$\frac{d}{dx} \frac{\log(C_m x(x + 1))}{x} = \frac{1 + \frac{x}{x+1} - \log(C_m x(x + 1))}{x^2} < 0.$$

Therefore,

$$\rho_{m,i}(t_r - 1) \leq c_m(r - 1), \quad r = 2, \dots, n.$$

For $\ell \geq 1$, recall

$$\Lambda_m(\ell) = \log \left(\frac{2Kj(m)(j(m) + 1)\ell(\ell + 1)}{\delta} \right).$$

Since $r - 1 \leq n - 1 \leq L_m$,

$$c_m(r - 1) \leq \sqrt{\frac{\Lambda_m(L_m)}{2(r - 1)}}.$$

Consequently,

$$\begin{aligned} \sum_{\substack{t \leq T: \\ m \in \mathcal{R}_t, a_t = i}} \rho_{m,i}(t - 1) &\leq \bar{P}_m + \sqrt{\frac{\Lambda_m(L_m)}{2}} \sum_{s=1}^{n-1} \frac{1}{\sqrt{s}} \\ &\leq \bar{P}_m + \sqrt{2\Lambda_m(L_m)n}, \end{aligned}$$

where we used

$$\sum_{s=1}^{n-1} \frac{1}{\sqrt{s}} \leq 2\sqrt{n}.$$

Thus, for every m with $L_m > 0$ and every arm i ,

$$\sum_{\substack{t \leq T: \\ m \in \mathcal{R}_t, a_t = i}} \rho_{m,i}(t - 1) \leq \min \left\{ N_{m,i}^R(T) \bar{P}_m, \bar{P}_m + \sqrt{2\Lambda_m(L_m)N_{m,i}^R(T)} \right\}.$$

We now sum over arms. Since

$$\sum_{i=1}^K N_{m,i}^R(T) = L_m,$$

the first bound gives

$$\sum_{i=1}^K \sum_{\substack{t \leq T: \\ m \in \mathcal{R}_t, a_t = i}} \rho_{m,i}(t-1) \leq L_m \bar{P}_m.$$

Alternatively,

$$\begin{aligned} \sum_{i=1}^K \sum_{\substack{t \leq T: \\ m \in \mathcal{R}_t, a_t = i}} \rho_{m,i}(t-1) &\leq K \bar{P}_m + \sqrt{2\Lambda_m(L_m)} \sum_{i=1}^K \sqrt{N_{m,i}^R(T)} \\ &\leq K \bar{P}_m + \sqrt{2K\Lambda_m(L_m)L_m}, \end{aligned}$$

where the last inequality follows from Cauchy–Schwarz. Taking the smaller of the two bounds yields

$$\sum_{i=1}^K \sum_{\substack{t \leq T: \\ m \in \mathcal{R}_t, a_t = i}} \rho_{m,i}(t-1) \leq \min \left\{ L_m \bar{P}_m, K \bar{P}_m + \sqrt{2K\Lambda_m(L_m)L_m} \right\}.$$

Summing over all agents with $L_m > 0$,

$$\sum_{t=1}^T B_t^{\text{stat}}(a_t) \leq \sum_{m: L_m > 0} \min \left\{ L_m \bar{P}_m, K \bar{P}_m + \sqrt{2K\Lambda_m(L_m)L_m} \right\}.$$

Finally,

$$\sum_{t=1}^T E_t^A = \sum_{t=1}^T \sum_{m \in \mathcal{A}_t} \bar{P}_m.$$

Substituting the last two bounds into the regret decomposition gives

$$R_T \leq 2 \sum_{t=1}^T \sum_{m \in \mathcal{A}_t} \bar{P}_m + 2 \sum_{m: L_m > 0} \min \left\{ L_m \bar{P}_m, K \bar{P}_m + \sqrt{2K\Lambda_m(L_m)L_m} \right\},$$

which proves the claim. $\square$

A.5.2 Proof of Theorem 2

Proof. We prove the two lower bounds separately.

Lower bound for newly arriving agents. Introduce one permanent agent m_0 whose rewards are identically zero on both arms and are known to the learner. At round t , let

$$\mathcal{M}_t = \{m_0\} \cup \mathcal{A}_t.$$

Every agent in $\mathcal{A}_t$ departs before round $t + 1$. Thus, each such agent is newly arriving at round t and is never a continuing agent.

For every sign vector

$$z = (z_1, \dots, z_T) \in \{-1, +1\}^T,$$

construct an instance ν^z as follows. For every $m \in \mathcal{A}_t$, set

$$(\mu_m^1, \mu_m^2) = \begin{cases} (p_m, 0), & z_t = +1, \\ (0, p_m), & z_t = -1. \end{cases}$$

Rewards are deterministic at these means. Since $p_m \in [0, 1]$, all reward distributions are supported on $[0, 1]$.

For every arriving agent, provide the same entry information under the two configurations:

$$\hat{\mu}_{m,1}^{\text{entry}} = \hat{\mu}_{m,2}^{\text{entry}} = 0, \quad \bar{P}_m = p_m.$$

The entry estimates lie in $[0, 1]$, and under either value of z_t ,

$$P_m = \max_{i \in \{1,2\}} |\hat{\mu}_{m,i}^{\text{entry}} - \mu_m^i| = p_m = \bar{P}_m.$$

Thus, the entry information available to the learner is identical under $z_t = +1$ and $z_t = -1$.

Let

$$S_t = \sum_{m \in \mathcal{A}_t} p_m.$$

Because the permanent agent is arm-indifferent, if $z_t = +1$, then

$$V_t(1) = S_t, \quad V_t(2) = 0,$$

whereas if $z_t = -1$, then

$$V_t(1) = 0, \quad V_t(2) = S_t.$$

Hence, whenever $S_t > 0$, the optimal arm is 1 under $z_t = +1$ and 2 under $z_t = -1$, and selecting the other arm incurs instantaneous regret exactly S_t . When $S_t = 0$, the instantaneous regret is zero, so the same calculation below remains valid.

Now let $Z_1, \dots, Z_T$ be independent Rademacher random variables, i.e.,

$$\mathbb{P}(Z_t = +1) = \mathbb{P}(Z_t = -1) = \frac{1}{2}.$$

This induces the uniform prior over the 2^T instances $\{\nu^z : z \in \{-1, +1\}^T\}$.

Fix any possibly randomized nonanticipating policy π . Let $\mathcal{F}_t^-$ denote the sigma-field containing all information available immediately before the policy selects a_t , including the complete past history, the current active population, and the entry information of all agents in $\mathcal{A}_t$.

By construction, the information in $\mathcal{F}_t^-$ is identical under $Z_t = +1$ and $Z_t = -1$. Moreover, Z_t is independent of $Z_1, \dots, Z_{t-1}$. Therefore,

$$Z_t \perp\!\!\!\perp \mathcal{F}_t^-.$$

Let

$$q_t = \mathbb{P}_\pi(a_t = 1 \mid \mathcal{F}_t^-).$$

Conditional on $\mathcal{F}_t^-$, when $Z_t = +1$, the expected instantaneous regret is

$$(1 - q_t)S_t,$$

whereas when $Z_t = -1$, it is

$$q_t S_t.$$

Averaging over Z_t gives

$$\mathbb{E}[V_t(i_t^*) - V_t(a_t) \mid \mathcal{F}_t^-] = \frac{1}{2}(1 - q_t)S_t + \frac{1}{2}q_t S_t = \frac{S_t}{2}.$$

Taking expectations and summing over t ,

$$\mathbb{E}_{Z, \pi}[R_T] = \frac{1}{2} \sum_{t=1}^T S_t = \frac{1}{2} \sum_{t=1}^T \sum_{m \in \mathcal{A}_t} p_m.$$

The left-hand side is the average of $\mathbb{E}_{\nu^z}^\pi[R_T]$ over the 2^T instances ν^z . Hence, there exists at least one $z \in \{-1, +1\}^T$ such that

$$\mathbb{E}_{\nu^z}^\pi[R_T] \geq \frac{1}{2} \sum_{t=1}^T \sum_{m \in \mathcal{A}_t} p_m.$$

Consequently,

$$\sup_{\nu \in \mathcal{I}(\{p_m\})} \mathbb{E}_\nu^\pi[R_T] \geq \frac{1}{2} \sum_{t=1}^T \sum_{m \in \mathcal{A}_t} p_m = \frac{1}{2} \sum_{t=1}^T \sum_{m \in \mathcal{A}_t} \bar{P}_m.$$

Lower bound for post-entry learning. We next construct a different family of instances. Consider any collection

$$\{(L_m, p_m) : L_m > 0\}, \quad L_m \in \mathbb{N}, \quad p_m \in [0, 1],$$

and define

$$\mathcal{H} = \{m : L_m > 0, p_m > 0\}.$$

Agents with $p_m = 0$ contribute zero to the desired lower bound and may be ignored in the hard construction. Suppose

$$\sum_{m \in \mathcal{H}} (L_m + 1) \leq T.$$

For every $m \in \mathcal{H}$, define

$$H_m = L_m + 1.$$

Fix any ordering of the agents in $\mathcal{H}$, and assign them pairwise disjoint consecutive blocks

$$\mathcal{B}_m = \{s_m, s_m + 1, \dots, s_m + H_m - 1\} \subseteq [T]$$

with

$$|\mathcal{B}_m| = H_m.$$

Agent m arrives at the first round of $\mathcal{B}_m$, remains active throughout the block, and departs immediately afterward. Thus, it is newly arriving during the first round of $\mathcal{B}_m$ and is a continuing agent during exactly L_m subsequent rounds.

For each $m \in \mathcal{H}$, introduce an auxiliary agent b_m that is active over the same block. Its reward means are known exactly to the learner and its pre-training error is zero. We also retain the permanent agent m_0 , whose rewards are identically zero on both arms. Hence, for $t \in \mathcal{B}_m$,

$$\mathcal{M}_t = \{m_0, m, b_m\},$$

whereas at rounds outside the blocks,

$$\mathcal{M}_t = \{m_0\}.$$

The latter rounds incur zero regret.

The auxiliary agents and m_0 have pre-training error zero and therefore contribute zero to

$$\min\{L_m p_m, \sqrt{K L_m}\}.$$

Thus, adding them does not affect the right-hand side of the desired lower bound.

Recall that $K = 2$. For every $m \in \mathcal{H}$, define

$$\varepsilon_m = \min \left\{ p_m, \frac{1}{16} \sqrt{\frac{K}{H_m}} \right\}.$$

Since $K = 2$ and $H_m \geq 1$,

$$0 < \varepsilon_m \leq \frac{\sqrt{2}}{16} < \frac{1}{8}.$$

For $p \in [0, 1]$, define

$$c_1(p) = \left(\frac{1}{2} - p \right)_+, \quad c_2(p) = \left(p - \frac{1}{2} \right)_+.$$

Then

$$c_1(p), c_2(p) \in \left[0, \frac{1}{2} \right]$$

and

$$p + c_1(p) = \frac{1}{2} + c_2(p) = \max \left\{ p, \frac{1}{2} \right\}.$$

For every collection of signs

$$z = \{z_m \in \{-1, +1\} : m \in \mathcal{H}\},$$

construct an instance ν^z as follows. During block $\mathcal{B}_m$, the hard agent m has mean rewards

$$\mu_m^1 = p_m, \quad \mu_m^2 = \frac{1}{2} + z_m \varepsilon_m.$$

More precisely, arm 1 gives the deterministic reward p_m , while arm 2 gives independent Bernoulli rewards with mean $\frac{1}{2} + z_m \varepsilon_m$. Since $\varepsilon_m < 1/8$, these rewards are supported on $[0, 1]$.

The auxiliary agent b_m has deterministic rewards with means

$$\mu_{b_m}^1 = c_1(p_m), \quad \mu_{b_m}^2 = c_2(p_m).$$

The permanent agent m_0 has deterministic reward zero on both arms.

For the hard agent m , provide the entry estimate

$$\hat{\mu}_{m,1}^{\text{entry}} = 0, \quad \hat{\mu}_{m,2}^{\text{entry}} = \frac{1}{2}, \quad \bar{P}_m = p_m.$$

Both coordinates belong to $[0, 1]$. Moreover,

$$\left| \hat{\mu}_{m,1}^{\text{entry}} - \mu_m^1 \right| = p_m$$

and

$$\left| \hat{\mu}_{m,2}^{\text{entry}} - \mu_m^2 \right| = \varepsilon_m \leq p_m.$$

Hence

$$P_m = \max_{a \in \{1,2\}} \left| \hat{\mu}_{m,a}^{\text{entry}} - \mu_m^a \right| = p_m = \bar{P}_m.$$

Thus, the prescribed pre-training error is attained exactly, while the entry information does not depend on z_m .

For the auxiliary agent, provide its exact means as its entry estimate:

$$\hat{\mu}_{b_m,1}^{\text{entry}} = c_1(p_m), \quad \hat{\mu}_{b_m,2}^{\text{entry}} = c_2(p_m), \quad \bar{P}_{b_m} = 0.$$

Therefore $P_{b_m} = 0$, and all entry estimates in the construction lie in $[0, 1]$.

Define

$$B_m = p_m + c_1(p_m) = \frac{1}{2} + c_2(p_m).$$

For every $t \in \mathcal{B}_m$, the aggregate mean rewards are

$$V_t(1) = p_m + c_1(p_m) = B_m$$

and

$$V_t(2) = \frac{1}{2} + z_m \varepsilon_m + c_2(p_m) = B_m + z_m \varepsilon_m.$$

Thus, when $z_m = +1$, arm 2 is uniquely optimal, whereas when $z_m = -1$, arm 1 is uniquely optimal. In either case, selecting the suboptimal arm incurs instantaneous aggregate regret exactly ε_m .

We now lower-bound the regret accumulated over one block. Fix $m \in \mathcal{H}$, and abbreviate

$$H = H_m, \quad \varepsilon = \varepsilon_m.$$

Condition on an arbitrary realization of the complete history before $\mathcal{B}_m$ begins. The policy may depend arbitrarily on this history. The sign z_m will be chosen independently of that history, so the following argument applies conditionally on every such realization.

Let $\mathbb{P}_+$ and $\mathbb{P}_-$ denote the conditional laws of the transcript generated during $\mathcal{B}_m$ under $z_m = +1$ and $z_m = -1$, respectively. Let

$$N_i = \sum_{t \in \mathcal{B}_m} \mathbf{1}\{a_t = i\}, \quad i \in \{1, 2\}.$$

Since exactly one common arm is selected at every round,

$$N_1 + N_2 = H$$

pathwise.

The two instances have identical populations, entry information, auxiliary rewards, arm-1 rewards of the hard agent, and action kernels of the policy. They differ only in the reward distribution of agent m when arm 2 is selected. Therefore, by the chain rule for relative entropy,

$$D_{\text{KL}}(\mathbb{P}_- \parallel \mathbb{P}_+) = \mathbb{E}_-[N_2] \text{kl}\left(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right),$$

where $\text{kl}(\cdot, \cdot)$ denotes Bernoulli relative entropy.

For $0 \leq \varepsilon \leq 1/8$,

$$\begin{aligned} \text{kl}\left(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right) &= 2\varepsilon \log\left(\frac{1+2\varepsilon}{1-2\varepsilon}\right) \\ &\leq 12\varepsilon^2. \end{aligned}$$

Indeed, for $0 \leq x \leq 1/4$,

$$\log\left(\frac{1+x}{1-x}\right) \leq \frac{2x}{1-x}.$$

Taking $x = 2\varepsilon$ gives

$$\begin{aligned} 2\varepsilon \log\left(\frac{1+2\varepsilon}{1-2\varepsilon}\right) &\leq \frac{8\varepsilon^2}{1-2\varepsilon} \\ &\leq \frac{32}{3}\varepsilon^2 \\ &\leq 12\varepsilon^2. \end{aligned}$$

Hence

$$D_{\text{KL}}(\mathbb{P}_- \parallel \mathbb{P}_+) \leq 12H\varepsilon^2.$$

By the definition of ε ,

$$H\varepsilon^2 \leq \frac{K}{256} = \frac{1}{128},$$

where the equality uses $K = 2$. Therefore,

$$D_{\text{KL}}(\mathbb{P}_- \parallel \mathbb{P}_+) \leq \frac{3}{32}.$$

Pinsker's inequality then gives

$$\|\mathbb{P}_- - \mathbb{P}_+\|_{\text{TV}} \leq \sqrt{\frac{1}{2}D_{\text{KL}}(\mathbb{P}_- \parallel \mathbb{P}_+)} \leq \frac{\sqrt{3}}{8} < \frac{1}{4}.$$

Let R_m denote the regret accumulated over block $\mathcal{B}_m$. Under $\mathbb{P}_+$, arm 2 is optimal, so every selection of arm 1 incurs regret ε . Thus,

$$\mathbb{E}_+[R_m] = \varepsilon\mathbb{E}_+[N_1].$$

Under $\mathbb{P}_-$, arm 1 is optimal, so every selection of arm 2 incurs regret ε . Thus,

$$\mathbb{E}_-[R_m] = \varepsilon \mathbb{E}_-[N_2].$$

Since $0 \leq N_1 \leq H$,

$$|\mathbb{E}_+[N_1] - \mathbb{E}_-[N_1]| \leq H \|\mathbb{P}_+ - \mathbb{P}_-\|_{\text{TV}}.$$

Therefore,

$$\mathbb{E}_+[N_1] \geq \mathbb{E}_-[N_1] - H \|\mathbb{P}_+ - \mathbb{P}_-\|_{\text{TV}}.$$

Using $N_1 + N_2 = H$ under $\mathbb{P}_-$,

$$\begin{aligned} \mathbb{E}_+[N_1] + \mathbb{E}_-[N_2] &\geq \mathbb{E}_-[N_1] + \mathbb{E}_-[N_2] - H \|\mathbb{P}_+ - \mathbb{P}_-\|_{\text{TV}} \\ &= H(1 - \|\mathbb{P}_+ - \mathbb{P}_-\|_{\text{TV}}) \geq \frac{3H}{4}. \end{aligned}$$

Consequently,

$$\frac{1}{2}(\mathbb{E}_+[R_m] + \mathbb{E}_-[R_m]) = \frac{\varepsilon}{2}(\mathbb{E}_+[N_1] + \mathbb{E}_-[N_2]) \geq \frac{3}{8}H\varepsilon.$$

Substituting the definition of ε_m ,

$$\frac{3}{8}H_m\varepsilon_m = \min \left\{ \frac{3}{8}H_m p_m, \frac{3}{128}\sqrt{KH_m} \right\}.$$

Since $H_m = L_m + 1 \geq L_m$,

$$\frac{3}{8}H_m p_m \geq \frac{1}{128}L_m p_m$$

and

$$\frac{3}{128}\sqrt{KH_m} \geq \frac{1}{128}\sqrt{KL_m}.$$

It follows that

$$\frac{1}{2}(\mathbb{E}_+[R_m] + \mathbb{E}_-[R_m]) \geq \frac{1}{128} \min \left\{ L_m p_m, \sqrt{KL_m} \right\}.$$

Finally, let

$$\{Z_m : m \in \mathcal{H}\}$$

be independent Rademacher random variables. Consider the resulting uniform prior over the finite family of instances

$$\{\nu^z : z = \{z_m \in \{-1, +1\} : m \in \mathcal{H}\}\}.$$

Before block $\mathcal{B}_m$ begins, Z_m is independent of the complete history generated by all blocks that precede $\mathcal{B}_m$. Moreover, the active population and all entry information

in $\mathcal{B}_m$ are identical under the two values of Z_m . Hence the preceding conditional two-instance argument applies after conditioning on the complete history before $\mathcal{B}_m$. Taking expectations gives

$$\mathbb{E}_{Z,\pi}[R_m] \geq \frac{1}{128} \min \left\{ L_m p_m, \sqrt{K L_m} \right\}$$

for every $m \in \mathcal{H}$.

Because the blocks are disjoint and all rounds outside them incur zero regret,

$$\mathbb{E}_{Z,\pi}[R_T] = \sum_{m \in \mathcal{H}} \mathbb{E}_{Z,\pi}[R_m] \geq \frac{1}{128} \sum_{m \in \mathcal{H}} \min \left\{ L_m p_m, \sqrt{K L_m} \right\}.$$

For every m with $L_m > 0$ but $p_m = 0$,

$$\min \left\{ L_m p_m, \sqrt{K L_m} \right\} = 0.$$

Therefore,

$$\mathbb{E}_{Z,\pi}[R_T] \geq \frac{1}{128} \sum_{m: L_m > 0} \min \left\{ L_m p_m, \sqrt{K L_m} \right\}.$$

The left-hand side is the average of $\mathbb{E}_{\nu^z}^\pi[R_T]$ over the finite family of instances $\{\nu^z\}$. Hence, at least one instance in the family satisfies

$$\mathbb{E}_{\nu^z}^\pi[R_T] \geq \frac{1}{128} \sum_{m: L_m > 0} \min \left\{ L_m p_m, \sqrt{K L_m} \right\}.$$

Consequently,

$$\sup_{\nu \in \mathcal{Q}(\{L_m, p_m\})} \mathbb{E}_\nu^\pi[R_T] \geq \frac{1}{128} \sum_{m: L_m > 0} \min \left\{ L_m p_m, \sqrt{K L_m} \right\}.$$

Both constructions apply to every admissible possibly randomized policy π , which proves the claims. $\square$

A.5.3 Proof of Theorem 3

Proof. By Assumption 1, for every $t \geq \tau_{\text{stab}}$ and $i \neq i^\dagger$,

$$V_t(i^\dagger) - V_t(i) = M_t(\bar{V}_t(i^\dagger) - \bar{V}_t(i)) \geq M_t \Delta.$$

By Lemma 9, with probability at least $1 - \delta/2$,

$$|\hat{V}_t(i) - V_t(i)| \leq B_t^{\text{stat}}(i) + E_t^A, \quad \forall t \in [T], i \in [K].$$

A union bound shows that both statements hold simultaneously with probability at least $1 - \delta$. We work on this event.

Fix $t \in \{\tau, \dots, T\}$ and $i \neq i^\dagger$. Since $t \geq \tau_{\text{stab}}$,

$$V_t(i^\dagger) - V_t(i) = M_t(\bar{V}_t(i^\dagger) - \bar{V}_t(i)) \geq M_t\Delta.$$

Since $t \geq \tau_{\text{id}}$,

$$W_t(i) \leq \frac{M_t\Delta}{4}.$$

Let

$$U_t(i) = \hat{V}_t(i) + B_t^{\text{stat}}(i)$$

be the Global-UCB index. On the simultaneous confidence event,

$$U_t(i^\dagger) \geq V_t(i^\dagger) - E_t^A,$$

whereas

$$U_t(i) \leq V_t(i) + 2B_t^{\text{stat}}(i) + E_t^A.$$

Therefore,

$$\begin{aligned} U_t(i^\dagger) - U_t(i) &\geq V_t(i^\dagger) - V_t(i) - 2B_t^{\text{stat}}(i) - 2E_t^A \\ &= V_t(i^\dagger) - V_t(i) - 2W_t(i) \\ &\geq M_t\Delta - \frac{M_t\Delta}{2} = \frac{M_t\Delta}{2} > 0. \end{aligned}$$

Thus $a_t = i^\dagger$. Since $t \geq \tau$ was arbitrary, the same conclusion holds for every subsequent round.

Under coordinated actions,

$$\bar{R}_T = \sum_{t=1}^T [\bar{V}_t(i_t^*) - \bar{V}_t(a_t)].$$

All terms from $t = \tau$ onward vanish, while each preceding term is at most one. Hence

$$\bar{R}_T \leq \tau - 1.$$

Similarly,

$$R_T = \sum_{t=1}^T [V_t(i_t^*) - V_t(a_t)] = \sum_{t=1}^T M_t [\bar{V}_t(i_t^*) - \bar{V}_t(a_t)],$$

so

$$R_T \leq \sum_{t=1}^{\tau-1} M_t.$$

□

A.5.4 Proof of Lemma 4

Proof. Fix an arm i , and order all agent-level observations collected from this arm chronologically, using an arbitrary fixed ordering among agents observed in the same round. Before the reward noises of a round are realized, the population, its representations, and the selected arm are already determined. Under the conditional independence assumption, this serialization produces a predictable sequence of kernel features with conditionally R -sub-Gaussian noise.

Applying the RKHS self-normalized concentration inequality with failure probability δ/K therefore gives, simultaneously over all sample sizes,

$$\left\| \widehat{f}_{i,t-1} - f_i \right\|_{\mathcal{A}_{i,t-1}} \leq S + R \sqrt{2 \left(\gamma_{N_i(t-1)} + 1 + \log \frac{K}{\delta} \right)}.$$

Since $\gamma_{N_i(t-1)} \leq \bar{\gamma}_{N_i(t-1)}$, the right-hand side is at most $\beta_{i,t-1}$. A union bound over the K arms gives a common event of probability at least $1 - \delta$.

On this event,

$$\begin{aligned} \left| \widehat{V}_t(i) - V_t(i) \right| &= \left| \left\langle \widehat{f}_{i,t-1} - f_i, \Psi_t \right\rangle_{\mathcal{H}_k} \right| \\ &\leq \left\| \widehat{f}_{i,t-1} - f_i \right\|_{\mathcal{A}_{i,t-1}} \|\Psi_t\|_{\mathcal{A}_{i,t-1}^{-1}} \\ &\leq \beta_{i,t-1} \sigma_{i,t}^{\text{pop}}, \end{aligned}$$

which proves the claim. $\square$

A.5.5 Proof of Theorem 5

Proof. Work on the event in Lemma 4. Let $i_t^* \in \arg \max_{i \in [K]} V_t(i)$. By the UCB rule,

$$\widehat{V}_t(i_t^*) + \beta_{i_t^*,t-1} \sigma_{i_t^*,t}^{\text{pop}} \leq \widehat{V}_t(a_t) + \beta_{a_t,t-1} \sigma_{a_t,t}^{\text{pop}}.$$

The confidence bounds therefore imply

$$V_t(i_t^*) - V_t(a_t) \leq 2\beta_{a_t,t-1} \sigma_{a_t,t}^{\text{pop}} \leq 2\beta_T \sigma_{a_t,t}^{\text{pop}}.$$

Thus,

$$R_T \leq 2\beta_T \sum_{t=1}^T \sigma_{a_t,t}^{\text{pop}}.$$

We next relate the population-functional uncertainty to the information gained from the batch of observations obtained at round t . Write $i = a_t$, let

$$\mathcal{B}_t = \sum_{m \in \mathcal{M}_t} \phi(x_m) \otimes \phi(x_m),$$

and define

$$\mathcal{X}_t = \mathcal{A}_{i,t-1}^{-1/2} \mathcal{B}_t \mathcal{A}_{i,t-1}^{-1/2}.$$

For $u_m = \mathcal{A}_{i,t-1}^{-1/2} \phi(x_m)$,

$$(\sigma_{a_t,t}^{\text{pop}})^2 = \left\| \sum_{m \in \mathcal{M}_t} u_m \right\|_{\mathcal{H}_k}^2 \leq M_t \sum_{m \in \mathcal{M}_t} \|u_m\|_{\mathcal{H}_k}^2 = M_t \text{tr}(\mathcal{X}_t).$$

Since $\mathcal{A}_{i,t-1} \succeq I$ and $k(x, x) \leq 1$,

$$\text{tr}(\mathcal{X}_t) = \sum_{m \in \mathcal{M}_t} \|u_m\|_{\mathcal{H}_k}^2 \leq M_t.$$

Hence every eigenvalue z of $\mathcal{X}_t$ lies in $[0, M_t]$. Because $z/\log(1+z)$ is increasing for $z \geq 0$,

$$\text{tr}(\mathcal{X}_t) \leq \frac{M_t}{\log(1+M_t)} \log \det(I + \mathcal{X}_t).$$

Consequently,

$$(\sigma_{a_t,t}^{\text{pop}})^2 \leq \frac{M_t^2}{\log(1+M_t)} \log \det(I + \mathcal{X}_t).$$

Let

$$\Delta \Gamma_t = \Gamma_{a_t,t} - \Gamma_{a_t,t-1}.$$

By the determinant identity for the update $\mathcal{A}_{a_t,t} = \mathcal{A}_{a_t,t-1} + \mathcal{B}_t$,

$$\Delta \Gamma_t = \frac{1}{2} \log \det(I + \mathcal{X}_t).$$

Therefore,

$$(\sigma_{a_t,t}^{\text{pop}})^2 \leq 2 \frac{M_t^2}{\log(1+M_t)} \Delta \Gamma_t.$$

By Cauchy–Schwarz,

$$\begin{aligned} \sum_{t=1}^T \sigma_{a_t,t}^{\text{pop}} &\leq \sqrt{2 \left(\sum_{t=1}^T \frac{M_t^2}{\log(1+M_t)} \right) \left(\sum_{t=1}^T \Delta \Gamma_t \right)} \\ &= \sqrt{2 H_T \Gamma_T^\Sigma}, \end{aligned}$$

because the information-gain increments telescope separately for each arm. Substitution proves the first claim.

Finally, arm i receives $N_i(T)$ agent-level observations, so

$$\Gamma_{i,T} \leq \gamma_{N_i(T)} \leq \bar{\gamma}_{N_i(T)}.$$

Summing over arms and using $N_i(T) \leq N_T$ proves the second claim. $\square$

A.5.6 Proof of Theorem 6

Proof. We first present the information gain under the kernel.

Lemma 10 (Information gain). *There exists a constant $C_\gamma > 0$, depending only on $(\mathcal{X}, g)$, ν , and ℓ , such that, for every $n \geq 1$, $\gamma_n \leq \bar{\gamma}_n = C_\gamma n^{\frac{d}{2\nu+d}} \log^{\frac{2\nu}{2\nu+d}}(e+n)$.*

Proof. Let

$$s = \nu + \frac{d}{2}.$$

Up to a positive constant depending only on the kernel parameters, the spectral coefficients of $k_{\nu,\ell}$ are

$$w_j = (a_\ell + \lambda_j)^{-s}.$$

We use two spectral estimates for the Laplace–Beltrami operator. First, the local Weyl law for the spectral function [Hörmander \[1968\]](#) implies that there exists $C_W > 0$ such that

$$\sup_{x \in \mathcal{X}} \sum_{\lambda_j \leq L} \varphi_j(x)^2 \leq C_W (1+L)^{d/2}, \quad L \geq 0.$$

Second, Weyl’s eigenvalue asymptotics for the Laplace–Beltrami operator [Hörmander \[1968\]](#), [Chavel \[1984\]](#) imply $\lambda_j \asymp j^{2/d}$ as $j \rightarrow \infty$. Hence, after adjusting the constant to include the finitely many low-frequency modes,

$$1 + \lambda_j \geq c_W (1+j)^{2/d}$$

for some $c_W > 0$ and all $j \geq 0$.

For $D \geq 1$, decompose

$$k_{\nu,\ell} = k_D + r_D,$$

where k_D contains the first D spectral modes. We first control the diagonal of the residual kernel. Let

$$L_D = a_\ell + \lambda_D.$$

Partition the indices $j \geq D$ into

$$\mathcal{J}_q = \{j \geq D : 2^q L_D \leq a_\ell + \lambda_j < 2^{q+1} L_D\}, \quad q \geq 0.$$

By the local Weyl bound for the spectral function of the Laplace–Beltrami operator Hörmander [1968],

$$\begin{aligned}
r_D(x, x) &= \sum_{j \geq D} w_j \varphi_j(x)^2 \\
&\leq C \sum_{q=0}^{\infty} (2^q L_D)^{-s} \sum_{j \in \mathcal{J}_q} \varphi_j(x)^2 \\
&\leq C' \sum_{q=0}^{\infty} (2^q L_D)^{-s} (2^{q+1} L_D)^{d/2} \\
&= C'' L_D^{-\nu} \sum_{q=0}^{\infty} 2^{-q\nu}.
\end{aligned}$$

Since $\nu > 0$, the geometric series converges. Weyl eigenvalue growth then gives

$$\sup_{x \in \mathcal{X}} r_D(x, x) \leq C_r D^{-2\nu/d}$$

for some $C_r > 0$.

Now fix arbitrary $x_1, \dots, x_n \in \mathcal{X}$, and decompose their Gram matrix as

$$K_n = K_{n,D} + R_{n,D}.$$

Both matrices are positive semidefinite, $\text{rank}(K_{n,D}) \leq D$, and, since $k_{\nu,\ell}(x, x) \leq 1$,

$$\text{tr}(K_{n,D}) \leq \text{tr}(K_n) \leq n.$$

Hence, by concavity of the logarithm,

$$\log \det(I_n + K_{n,D}) \leq D \log \left(1 + \frac{n}{D} \right).$$

For the residual term,

$$\begin{aligned}
\log \det(I_n + K_n) &= \log \det(I_n + K_{n,D}) \\
&\quad + \log \det \left(I_n + (I_n + K_{n,D})^{-1/2} R_{n,D} (I_n + K_{n,D})^{-1/2} \right) \\
&\leq \log \det(I_n + K_{n,D}) + \text{tr}(R_{n,D}) \\
&\leq D \log \left(1 + \frac{n}{D} \right) + C_r n D^{-2\nu/d}.
\end{aligned}$$

Therefore,

$$\gamma_n \leq \frac{1}{2} \left[D \log \left(1 + \frac{n}{D} \right) + C_r n D^{-2\nu/d} \right].$$

Set

$$p = \frac{d}{2\nu + d}, \quad q = \frac{2\nu}{2\nu + d} = 1 - p,$$

and, for $n \geq 2$, choose

$$D = \left\lceil \left(\frac{n}{\log(e + n)} \right)^p \right\rceil.$$

Then

$$D \log \left(1 + \frac{n}{D} \right) \leq C n^p \log^q(e + n),$$

while

$$n D^{-2\nu/d} \leq C' n^p \log^q(e + n).$$

Thus

$$\gamma_n \leq C_\gamma n^p \log^q(e + n).$$

The case $n = 1$ is absorbed by increasing C_γ , which completes the proof. $\square$

By Lemma 10,

$$\Gamma_{i,T} \leq \gamma_{N_i(T)} \leq C_\gamma N_i(T)^p \log^q(e + N_i(T)) \leq C_\gamma N_i(T)^p L_T^q.$$

Since $\sum_{i=1}^K N_i(T) = N_T$ and $p \in (0, 1)$, concavity gives

$$\sum_{i=1}^K N_i(T)^p \leq K^{1-p} N_T^p.$$

Hence

$$\Gamma_T^\Sigma \leq C_\gamma K^{1-p} N_T^p L_T^q.$$

Because $1 - p = q = 2\nu/(2\nu + d)$,

$$\Gamma_T^\Sigma \leq C_\gamma K^{\frac{2\nu}{2\nu+d}} N_T^{\frac{d}{2\nu+d}} L_T^{\frac{2\nu}{2\nu+d}}.$$

Likewise,

$$\beta_{i,t-1} \leq S + R \sqrt{2 \left[C_\gamma N_T^p L_T^q + 1 + \log \frac{K}{\delta} \right]} = \bar{\beta}_T.$$

Substituting these two bounds into Theorem 5 proves the result. $\square$

A.5.7 Proof of Theorem 7

Proof. Let

$$\Delta_t(i) = \frac{V_t(i_t^*) - V_t(i)}{M_t},$$

and let $\mathcal{P}_t^*$ denote the exploration matching at round t . Since $M_t - e_t$ agents select the leader h_t , while the e_t remaining agents are matched one-to-one with challengers,

$$R_T = \sum_{t=1}^T (M_t - e_t) \Delta_t(h_t) + \sum_{t=1}^T R_t^{\text{exp}},$$

where

$$R_t^{\text{exp}} = \sum_{(m,i) \in \mathcal{P}_t^*} \Delta_t(i).$$

Let $\mathcal{E}$ denote the simultaneous confidence event

$$\left| \widehat{V}_t(i) - V_t(i) \right| \leq \beta_{i,t-1} \sigma_{i,t}^{\text{pop}}, \quad \forall i \in [K], t \in [T].$$

By the population-functional confidence result, $\mathbb{P}(\mathcal{E}) \geq 1 - \delta$. We work on $\mathcal{E}$. Since $N_i(t-1) \leq N_T$ and $\bar{\gamma}_n$ is nondecreasing,

$$\beta_{i,t-1} \leq \beta_T$$

for every i and t .

Because $h_t \in \arg \max_i U_t(i)$,

$$U_t(h_t) \geq U_t(i_t^*) \geq V_t(i_t^*).$$

On $\mathcal{E}$,

$$U_t(h_t) = \widehat{V}_t(h_t) + \beta_{h_t,t-1} \sigma_{h_t,t}^{\text{pop}} \leq V_t(h_t) + 2\beta_{h_t,t-1} \sigma_{h_t,t}^{\text{pop}}.$$

Therefore,

$$V_t(i_t^*) - V_t(h_t) \leq 2\beta_T \sigma_{h_t,t}^{\text{pop}},$$

and hence

$$(M_t - e_t) \Delta_t(h_t) \leq 2\beta_T \frac{M_t - e_t}{M_t} \sigma_{h_t,t}^{\text{pop}}.$$

Define the embedding of the agents selecting the leader by

$$\Psi_t^h = \sum_{m \in \mathcal{Q}_{t,h_t}} \phi(x_m).$$

Since $\Psi_t = \Psi_t^h + (\Psi_t - \Psi_t^h)$, the triangle inequality gives

$$\sigma_{h_t,t}^{\text{pop}} \leq \|\Psi_t^h\|_{\mathcal{A}_{h_t,t-1}^{-1}} + \|\Psi_t - \Psi_t^h\|_{\mathcal{A}_{h_t,t-1}^{-1}}.$$

Thus

$$\begin{aligned} (M_t - e_t)\Delta_t(h_t) &\leq 2\beta_T \frac{M_t - e_t}{M_t} \|\Psi_t^h\|_{\mathcal{A}_{h_t,t-1}^{-1}} \\ &\quad + 2\beta_T \frac{M_t - e_t}{M_t} \|\Psi_t - \Psi_t^h\|_{\mathcal{A}_{h_t,t-1}^{-1}}. \end{aligned}$$

We first control the uncertainty contributed by the leader batch. Let

$$A_t = \mathcal{A}_{h_t,t-1}, \quad z_m = A_t^{-1/2} \phi(x_m), \quad m \in \mathcal{Q}_{t,h_t},$$

and

$$C_t = \sum_{m \in \mathcal{Q}_{t,h_t}} z_m \otimes z_m.$$

Since $|\mathcal{Q}_{t,h_t}| = M_t - e_t$,

$$\begin{aligned} \|\Psi_t^h\|_{A_t^{-1}}^2 &= \left\| \sum_{m \in \mathcal{Q}_{t,h_t}} z_m \right\|_{\mathcal{H}_k}^2 \\ &\leq (M_t - e_t) \sum_{m \in \mathcal{Q}_{t,h_t}} \|z_m\|_{\mathcal{H}_k}^2 \\ &= (M_t - e_t) \text{tr}(C_t). \end{aligned}$$

Because $A_t \succeq I$ and $k(x, x) \leq 1$,

$$\text{tr}(C_t) = \sum_{m \in \mathcal{Q}_{t,h_t}} \langle \phi(x_m), A_t^{-1} \phi(x_m) \rangle_{\mathcal{H}_k} \leq M_t - e_t.$$

Hence every eigenvalue λ of C_t lies in $[0, M_t - e_t]$. Since $z/\log(1+z)$ is nondecreasing on $[0, \infty)$,

$$\lambda \leq \frac{M_t - e_t}{\log(1 + M_t - e_t)} \log(1 + \lambda).$$

Summing over the eigenvalues gives

$$\text{tr}(C_t) \leq \frac{M_t - e_t}{\log(1 + M_t - e_t)} \log \det(I + C_t).$$

Define the information-gain increment of the leader batch by

$$\Delta \Gamma_t^h = \frac{1}{2} \log \det(I + C_t).$$

By the determinant identity for the batch update of arm h_t , this is exactly the increase in its realized information gain produced by the $M_t - e_t$ leader observations at round t . Therefore,

$$\|\Psi_t^h\|_{A_t^{-1}}^2 \leq \frac{2(M_t - e_t)^2}{\log(1 + M_t - e_t)} \Delta\Gamma_t^h,$$

and consequently

$$\left(\frac{M_t - e_t}{M_t} \|\Psi_t^h\|_{A_t^{-1}} \right)^2 \leq 2 \frac{(M_t - e_t)^4}{M_t^2 \log(1 + M_t - e_t)} \Delta\Gamma_t^h.$$

Cauchy–Schwarz now yields

$$\sum_{t=1}^T \frac{M_t - e_t}{M_t} \|\Psi_t^h\|_{A_t^{-1}} \leq \sqrt{2H_T^{\text{het}} \sum_{t=1}^T \Delta\Gamma_t^h}.$$

For each arm i , the information increments obtained on rounds for which $h_t = i$ are a subset of the nonnegative information increments accumulated by that arm. Hence

$$\sum_{t=1}^T \Delta\Gamma_t^h \leq \sum_{i=1}^K \Gamma_{i,T} = \Gamma_T^\Sigma.$$

It follows that

$$\sum_{t=1}^T (M_t - e_t) \Delta_t(h_t) \leq 2\beta_T \sqrt{2\Gamma_T^\Sigma H_T^{\text{het}}} + 2\beta_T D_T,$$

where, for this proof only, we write

$$D_T = \sum_{t=1}^T \frac{M_t - e_t}{M_t} \|\Psi_t - \Psi_t^h\|_{A_{h_t, t-1}^{-1}}.$$

Combining this inequality with the regret decomposition gives

$$R_T \leq 2\beta_T \sqrt{2\Gamma_T^\Sigma H_T^{\text{het}}} + 2\beta_T D_T + \sum_{t=1}^T R_t^{\text{exp}}.$$

We next eliminate the two auxiliary terms. Since exactly e_t agents do not select the leader,

$$\Psi_t - \Psi_t^h = \sum_{m \in \mathcal{M}_t \setminus \mathcal{Q}_{t, h_t}} \phi(x_m).$$

Moreover, $\mathcal{A}_{h_t, t-1}^{-1} \preceq I$ and $\|\phi(x_m)\|_{\mathcal{H}_k}^2 = k(x_m, x_m) \leq 1$, so

$$\|\Psi_t - \Psi_t^h\|_{\mathcal{A}_{h_t, t-1}^{-1}} \leq \sum_{m \in \mathcal{M}_t \setminus \mathcal{Q}_{t, h_t}} \|\phi(x_m)\|_{\mathcal{A}_{h_t, t-1}^{-1}} \leq e_t.$$

Therefore,

$$D_T \leq \sum_{t=1}^T \frac{M_t - e_t}{M_t} e_t \leq N_T^{\text{exp}}.$$

Also, because $f_i(x) \in [0, 1]$, each population-average reward belongs to $[0, 1]$, and hence

$$0 \leq \Delta_t(i) \leq 1.$$

Since $|\mathcal{P}_t^*| = e_t$,

$$R_t^{\text{exp}} = \sum_{(m, i) \in \mathcal{P}_t^*} \Delta_t(i) \leq e_t.$$

Thus,

$$\sum_{t=1}^T R_t^{\text{exp}} \leq N_T^{\text{exp}},$$

which proves

$$R_T \leq 2\beta_T \sqrt{2\Gamma_T^\Sigma H_T^{\text{het}}} + (2\beta_T + 1)N_T^{\text{exp}}.$$

We now prove the second bound. Suppose that every exploratory assignment $(m, i) \in \mathcal{P}_t^*$ satisfies

$$I_{i, t}(m) = \frac{1}{2} \log \left(1 + \left\langle \phi(x_m), \mathcal{A}_{i, t-1}^{-1} \phi(x_m) \right\rangle_{\mathcal{H}_k} \right) \geq \eta.$$

Because $\mathcal{P}_t^*$ is a matching, each challenger is assigned to at most one agent at round t . Thus, if $(m, i) \in \mathcal{P}_t^*$, arm $i \neq h_t$ receives exactly one observation at that round and its design operator is updated as

$$\mathcal{A}_{i, t} = \mathcal{A}_{i, t-1} + \phi(x_m) \otimes \phi(x_m).$$

By the matrix determinant lemma,

$$\Gamma_{i, t} - \Gamma_{i, t-1} = \frac{1}{2} \log \left(1 + \left\langle \phi(x_m), \mathcal{A}_{i, t-1}^{-1} \phi(x_m) \right\rangle_{\mathcal{H}_k} \right) = I_{i, t}(m).$$

Hence the total information gain contributed by exploratory assignments is

$$\sum_{t=1}^T \sum_{(m, i) \in \mathcal{P}_t^*} I_{i, t}(m).$$

These are nonnegative information-gain increments contained in the total increments accumulated across all arms. Therefore,

$$\sum_{t=1}^T \sum_{(m,i) \in \mathcal{P}_t^*} I_{i,t}(m) \leq \Gamma_T^\Sigma.$$

On the other hand, the threshold condition gives

$$\sum_{t=1}^T \sum_{(m,i) \in \mathcal{P}_t^*} I_{i,t}(m) \geq \eta \sum_{t=1}^T |\mathcal{P}_t^*| = \eta N_T^{\text{exp}}.$$

Consequently,

$$N_T^{\text{exp}} \leq \frac{\Gamma_T^\Sigma}{\eta}.$$

Substituting this inequality into the first regret bound yields

$$R_T \leq 2\beta_T \sqrt{2\Gamma_T^\Sigma H_T^{\text{het}}} + \frac{2\beta_T + 1}{\eta} \Gamma_T^\Sigma.$$

Finally, arm i receives $N_i(T)$ observations in total, so

$$\Gamma_{i,T} \leq \gamma_{N_i(T)} \leq \bar{\gamma}_{N_i(T)} \leq \bar{\gamma}_{N_T},$$

where the final inequality uses $N_i(T) \leq N_T$ and the monotonicity of $\bar{\gamma}_n$. Summing over $i \in [K]$ gives

$$\Gamma_T^\Sigma \leq K \bar{\gamma}_{N_T}.$$

This completes the proof. □